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FOR

MOSES ISRAEL FOLORUNSO

(PKA EASY NATION)

AS FUMSSA PRESIDENT

BUSINESS MATHEMATICS (FMS 102)

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MATRIX OPERATIONS

Introduction

Matrix operations are a set of rules governing the addition, subtraction and multiplication of matrices. These rules are important in dealing with the application of matrices in solving a number of problems like solutions to systems of linear equations, input-output analysis, etc.

Objectives

The essence of this session is to enable the reader to:

a) Solve problems involving matrix addition.
b) Solve problems involving matrix subtraction.
c) Solve problems involving matrix multiplication.

Main Text

Addition of Matrices

The addition rule for matrices states that two or more matrices can be added together provided that they are of the same size (or order). That is, a 2 x 2 matrix can only be added to a 2 x 2 matrix, a 3 x 3 matrix to a 3 x 3 matrix etc.

Illustrative example 1:

Suppose $A=\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$; $B=\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$; $C=\begin{pmatrix} 2 & 7 \\ 8 & 4 \\ 6 & 5 \end{pmatrix}$; $D=\begin{pmatrix} 3 & 9 \\ 8 & 6 \\ 7 & 5 \end{pmatrix}$

In the above examples, matrix A can only be added to matrix B, while matrix C can only be added to matrix D because they are of the same sizes. Thus:

$$A+B=\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}+\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}=\begin{pmatrix} 1+2 & 4+2 \\ -1+3 & 3+5 \end{pmatrix}=\begin{pmatrix} 3 & 6 \\ 2 & 8 \end{pmatrix}$$

$$C+D=\begin{pmatrix} 2 & 7 \\ 8 & 4 \\ 6 & 5 \end{pmatrix}+\begin{pmatrix} 3 & 9 \\ 8 & 6 \\ 7 & 5 \end{pmatrix}=\begin{pmatrix} 2+3 & 7+9 \\ 8+8 & 4+6 \\ 6+7 & 5+5 \end{pmatrix}=\begin{pmatrix} 5 & 16 \\ 16 & 10 \\ 13 & 10 \end{pmatrix}$$

Subtraction of Matrices

The rule for the subtraction of matrices states that two matrices can be subtracted from one another only if they are of the same size.

Illustrative example 2:

Using matrices A, B, C and D above, find

a) A-B b) B-A c) D-C

Solution

a) $A - B = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2-1 & 4-2 \\ -1-3 & 3-5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -4 & -2 \end{pmatrix}$

b) $B - A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1-2 & 2-4 \\ 3-(-1) & 5-3 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 4 & 2 \end{pmatrix}$

c) $D - C = \begin{pmatrix} 3 & 9 \\ 8 & 6 \\ 7 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 8 & 4 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 3-2 & 9-7 \\ 8-8 & 6-4 \\ 7-6 & 5-5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$

Multiplication of Matrices

The multiplicative rule for matrices states that the matrix A can be multiplied by the matrix B in the order AB provided that the number of columns in A is the same as the number of rows of B. Thus, if matrix A is of the order m x n and matrix B is of the order n x r, then the matrix of the product of A and B in the order AB will be of the order m x r.

Illustrative example 3:

Suppose $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

Matrix A is of the size 2 x 2, while matrix B is a 2 x 3 matrix. Since the number of columns in A is equal to the number of rows in B, then matrix A can be multiplied by matrix B in the order AB.

$$\text{Thus AB} = A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 2\times1+4\times3 & 2\times2+4\times4 & 2\times2+4\times5 \\ 1\times1+3\times3 & 1\times2+3\times4 & 1\times2+3\times5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+12 & 4+16 & 4+20 \\ 1+9 & 2+12 & 2+15 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 20 & 24 \\ 10 & 14 & 17 \end{pmatrix}$$

Note that the size of AB, which is the product of A and B in the order AB, is 2 x 3, taking the number of rows of A and the number of columns of B. Also note that it is not possible to obtain the product of A and B in the order BA while it is possible to multiply A and B in the order AB. It is not possible to do the same in the order BA. In matrices, therefore, AB $\neq$ BA.

In-Text Questions (ITQ)

Explain the multiplication of matrices.

In-Text Answer (ITA)

Matrix A can only be multiplied by matrix B if only the number of columns in A is equal to the number of rows in B.

Summary: This section discussed addition, subtraction and multiplication of matrices and it solved problems on each.

SELF-ASSESSMENT QUESTIONS (SAQ 1)

Given that:

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 7 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 4 \\ 2 & 4 & 2 \end{pmatrix}; \quad C = \begin{pmatrix} 3 & 6 \\ 2 & 4 \\ 2 & 8 \end{pmatrix}; \quad D = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}$$

Find

1. A + C
2. AD
3. BA
4. C-A

SOLUTION TO SYSTEM OF LINEAR EQUATIONS USING DETERMINANTS AND MATRICES

Introduction

Apart from the substitution and elimination methods used in solving simultaneous linear equations, determinants and matrices can also be used to solve systems of linear equations among other things.

Objectives

At the end of this session, the reader should be able to:

a) Solve systems of linear equations using determinants.
b) Solve systems of linear equations using matrices.

Main Text

Solution to systems of linear equations using determinants.
For a system of linear equations in 2 unknown variables as in:

$$a_1 x + b_1 y = c_1 \text{ ....(1)}$$
$$a_2 x + b_2 y = c_2 \text{ ....(2)}$$

The unknown variables in equations (1) and (2) are x and y, while *a* and *b* are the coefficients of *x* and *y* respectively, C_1 and C_2 are constants of equations (1) and (2). Since equations (1) and (2) are simultaneous equations, *x* and *y* can be solved for either by way of substitution or elimination. To eliminate *y* for instance, we multiply equations (1) by b_2 and (2) by b_1. That is:

$$a_1 x + b_1 y = c_1 \text{ ....}(1) \times b_2$$
$$a_2 x + b_2 y = c_2 \text{ ....} (2) \times b_1$$

Then we have

$$a_1 b_2 x + b_1 b_2 y = b_2 c_1 \text{ ....(3)}$$
$$a_2 b_1 x + b_1 b_2 y = b_1 c_2 \text{ ....(4)}$$

Now, subtracting equation (4) from (3), we have

$$a_1 b_2 x - a_2 b_1 x + b_1 b_2 y - b_1 b_2 y = b_2 c_1 - b_1 c_2$$

This becomes

$$a_1 b_2 x - a_2 b_1 x = b_2 c_1 - b_1 c_2$$

Since *x* is common on the left hand side of the equation, we can factorise it as follows:

$$x(a_1 b_2 - a_2 b_1) = b_2 c_1 - b_1 c_2$$

$$x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} \text{.........................} (5)$$

In the form of determinants, the above equation for x can be written as:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{..................} (6)$$

If we had eliminated x, then we would have found the value of y. To eliminate x, we multiply equation (1) by a_2 and equation (2) by a_2 . The result will be as follows:

$$a_1 a_2 x + a_2 b_1 y = a_2 c_1 \text{................} (7)$$

$$a_1 a_2 x + a_1 b_2 y = a_1 c_2 \text{................} (8)$$

Now, subtract equation (7) from equation (8), i.e.:

$$a_1 a_2 x - a_1 a_1 x + a_1 b_2 y - a_2 b_1 y = a_1 c_2 - a_2 c_1$$

The above equation becomes:

$$a_1 b_2 x - a_2 b_1 y = a_1 c_2 - a_2 c_1$$

Factorizing the left hand side, we have:

$$y(a_1 b_2 - a_2 b_1) = a_1 c_2 - a_2 c_1$$

$$y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \text{........................} (9)$$

Again, in the form of determinants, equation (9) can be written as follows:

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} (10)$$

Thus, to solve for a system of linear equations involving 2 unknown variables, we use the determinant arrangements as given in equation (6) for the first unknown variable and equation (10) for the second unknown variable.

Illustrative example 1:

Suppose we have the following simultaneous equations:

$$4x - y = 10$$

$$x + 2y = 7$$

The values of the unknown variables x and y can be found using the determinant arrangements in equations (6) and (10), respectively.

$$x = \frac{\begin{vmatrix} 10 & -1 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix}}$$

$$x = \frac{(10\times2) - (7\times-1)}{(4\times2) - (1\times-1)}$$

$$x = \frac{20 - (-7)}{8 - (-1)}$$

$$x = \frac{20+7}{8+1}$$

$$x = \frac{27}{9}$$

$$x = 3$$

And

$$y = \frac{\begin{vmatrix} 4 & 10 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$y = \frac{(4\times7) - (1\times10)}{(4\times2) - (1\times-1)}$$

$$y = \frac{28 - 10}{8 - (-1)}$$

$$y = \frac{18}{8 + 1}$$

$$y = \frac{18}{9}$$

$$X = 2$$

Thus, the value of $x = 3$, and $y = 2$.
Similar determinant arrangements are also used in solving systems of linear equations involving 3 unknown variables.

Suppose a system of linear equations in 3 unknown variables is given as follows:

$a_1 x + b_1 y + c_1 z = d_1$...........................(1)
$a_2 x + b_2 y + c_2 z = d_2$..........................(2)
$a_3 x + b_3 y + c_3 z = d_3$..........................(3)

Where:

a_1, a_2 *and* a_3 b_1, b_2 *and* b_3; *and* c_1, c_2 *and* c_3 are the coefficients of the unknown variables *x*, *y* and *z*, respectively, while d_1, d_2 *and* d_3 are the constants of equations 1, 2 and 3 respectively.

The values of the variables can be obtained as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}; \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}; \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Illustrative example 2
Consider the following system of linear equations involving three unknown variables:

$$x - y + 2z = 3$$

$$2x + 3y - z = 11$$

$$x + 2y + z = 8$$

The values of the unknown variables can be found as follows:

$$x = \frac{\begin{vmatrix} 3 & -1 & 2 \\ 11 & 3 & -1 \\ 8 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{vmatrix}}$$

Solving the determinants using diagonals method we have:

$$x = \frac{\left|\begin{array}{ccc|cc} 3 & -1 & 2 & 3 & -1 \\ 11 & 3 & -1 & 11 & 3 \\ 8 & 2 & 1 & 8 & 2 \end{array}\right.}{\left|\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 & 3 \\ 1 & 2 & 1 & 1 & 2 \end{array}\right.}$$

$$x = \frac{(3\times3\times1) + (-1x\ -1\times8) + (2\times11\times2) - (8\times3\times2) + (2x\ -1\times3) + (1\times11\ x - -1)}{(1\times3\times1) + (-1x\ -1\times1) + (2\times2\times2) - (1\times3\times2) + (2x\ -1\times1) + (1\times2x\ -1)}$$

$$x = \frac{(9+8+44)-(48-611)}{(3+1+8)-(6-2-2)}$$

$$x = \frac{61-31}{12-2}$$

$$x = \frac{30}{10}$$

$$x = 3$$

$$y = \frac{\left|\begin{array}{ccc|cc} 1 & 3 & 2 & 1 & 3 \\ 2 & 11 & -1 & 2 & 11 \\ 1 & 8 & 1 & 1 & 8 \end{array}\right|}{\left|\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 & 3 \\ 1 & 2 & 1 & 1 & 2 \end{array}\right|}$$

$$y = \frac{(1\times1\times1)+(3x-1\times1)+(2\times2\times8)-(1\times1\times2)+(8x-1\times1)+(1\times2\times3)}{(1\times3\times1)+(-1x-1\times1)+(2\times2\times2)-(1\times3\times2)+(2x-1\times1)+(1\times2x-1)}$$

$$y = \frac{(11-3+32)-(22-8+6)}{(3+1+8)-(6-2-2)}$$

$$y = \frac{40-20}{12-2}$$

$$y = \frac{20}{10}$$

$$y = 2$$

$$z = \frac{\left|\begin{array}{ccc|cc} 1 & 1 & 3 & 1 & -1 \\ 2 & 3 & 11 & 2 & 3 \\ 1 & 2 & 8 & 1 & 2 \end{array}\right|}{\left|\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 & 3 \\ 1 & 2 & 1 & 1 & 2 \end{array}\right|}$$

$$z = \frac{(1\times3\times8)+(-1\times1\times1)+(3\times2\times2)-(1\times3\times3)+(2\times1\times1)+(8\times2\times-1)}{(1\times3\times1)+(-1x-1\times1)+(2\times2\times2)-(1\times3\times2)+(2x-1\times1)+(1\times2x-1)}$$

$$z = \frac{(24-11+12)-(9-22+16)}{(3+1+8)-(6-2-2)}$$

$$z = \frac{25-15}{12-2}$$

$$z = \frac{10}{10}$$

$$z = 1$$

Thus, $x = 3$; $y = 2$; $z = 1$

Like determinants, matrices can also be used in solving systems of linear equations. Suppose we refer to the system of linear equations in two unknown variables as given in equations (1) and (2). That is:

$$a_1 x + b_1 y = c_1 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$a_2 x + b_2 y = c_2 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From equations (1) and (2), the matrix of coefficients, A will be:

$$a = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

The matrix of unknown variables, x will be:

$$x = \begin{pmatrix} x \\ y \end{pmatrix}$$

And the matrix of constants, b will be:

$$b = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In the form of matrices, the above system of linear equations in equations (1) and (2) can be written as follows:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The above matrices have been defined as a, x and b. Thus, the above matrix relationship can be written as:

$$ax = b$$

Since x is the matrix of unknown variables, we can find the values of the unknown variables by making x the subject of the formula.

That is:

$$x = \frac{b}{a}$$

$$x = a^{-1}b$$

or

Solutions to systems of linear equations involving any number of variables can be obtained using matrices by solving for *x* in the above formula. And since *a* is a matrix, a^{-1} is the inverse of matrix *a*, which is given by the formula:

$$a^{-1} = \frac{1}{|a|} \times adjo\text{ int } of\ \mathbf{a}$$

Illustrative example 3

Recall the system of linear equations in example 1. That is:

$$4x - y = 10$$
$$x + 2y = 7$$

Find the values of *x* and *y* using matrices.

Solution

From the above equations, the matrix of coefficients ***a*** is given by:

$$a = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$$

The matrix of unknown variables, ***x*** is:

$$x = \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix of constants, ***b*** is:

$$b = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

Since solution is given by the formula $x = a^{-1}b$, we first find the inverse of matrix ***a*** which is given by:

$$a^{-1} = \frac{1}{|a|} \times Adjo\text{ int } of\ \mathbf{a}$$

$$|A| = \begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix} = (4 \times 2) - (1x - 2) = 8 - (-1) = 8 + 1 = 9$$

$$Cofactor\ of\ A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

$$\textit{Cofactor of } A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

$$\therefore \textit{ Adjoin of } A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

Hence, $A^{-1} = \frac{1}{9} \times \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} \frac{2}{9} & \frac{1}{9} \\ \frac{-1}{9} & \frac{4}{9} \end{pmatrix}$$

But $X = A^{-1}b$

That is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & \frac{1}{9} \\ \frac{-1}{9} & \frac{4}{9} \end{pmatrix} \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{9} \times 10 + \frac{1}{9} \times 7 \\ \frac{-1}{9} \times 10 + \frac{4}{9} \times 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{20}{9} + \frac{7}{9} \\ \frac{10}{9} + \frac{28}{9} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{27}{9} \\ \frac{18}{9} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\therefore X = 3; y = 2$

Illustrative example 4

Given the following system of linear equations, find the values of the unknown variables by way of matrices.

$$x - y + 2z = 3$$
$$2x + 3y - z = 11$$
$$x + 2y + z = 8$$

Solution

From the above system of linear equations:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; b = \begin{pmatrix} 3 \\ 11 \\ 8 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

To find the value of $|A|$ using the diagonals method, we have:

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & -1 \\ 2 & 3 \\ 1 & 2 \end{matrix}$$

=(1×3×1)+(-1× -1)+(2×2×2) –(1×3×2)+(2× -1×1)+(1×2× -1)
= 3+4+8–(6–2–2)
= 12–2
= 10

$$Cofactor\ of\ A = \begin{pmatrix} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \end{pmatrix}$$

$$Cofactor\ of\ \ A = \begin{pmatrix} 5 & 3 & 1 \\ -5 & -1 & 3 \\ -5 & -5 & 5 \end{pmatrix}$$

$$Cofactor\ of\ \ A = \begin{pmatrix} 5 & -3 & 1 \\ 5 & -3 & -3 \\ -5 & -5 & 5 \end{pmatrix}$$

$$\therefore\ Adjo\text{ int } of\ \ A = \begin{pmatrix} 5 & 5 & -5 \\ -3 & -1 & 5 \\ 1 & -3 & 5 \end{pmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{10} \times \begin{pmatrix} 5 & 5 & -5 \\ -3 & -1 & 5 \\ 1 & -3 & 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{5}{10} & \frac{5}{10} & \frac{5}{10} \\ \frac{3}{10} & \frac{-1}{10} & \frac{5}{10} \\ \frac{1}{10} & \frac{-3}{10} & \frac{5}{10} \end{pmatrix}$$

But $X = A^{-1}b$

That is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{10} & \frac{5}{10} & \frac{5}{10} \\ \frac{-3}{10} & \frac{1}{10} & \frac{5}{10} \\ \frac{1}{10} & \frac{-3}{10} & \frac{5}{10} \end{pmatrix} \begin{pmatrix} 3 \\ 11 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{10} \times 3 + \frac{5}{10} \times 11 + \frac{5}{10} \times 8 \\ \frac{-3}{10} \times 3 + \frac{1}{10} \times 11 + \frac{5}{10} \times 8 \\ \frac{1}{10} \times 3 + \frac{-3}{10} \times 11 + \frac{5}{10} \times 8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{15}{10} + \frac{55}{10} + \frac{40}{10} \\ \frac{-9}{10} + \frac{11}{10} + \frac{10}{10} \\ \frac{3}{10} + \frac{-33}{10} + \frac{10}{10} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{30}{10} \\ \frac{20}{10} \\ \frac{10}{10} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Hence x = 3; y = 2; z = 1

In-Text Questions (ITQ)

Given, $2x + 3y = 12$

$x - 2y = -1$

provide the matrix of coefficient, matrix of constant and matrix of unknown variables.

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In-Text Answer (ITA)

i. Matrix of coefficient are 2,3,1 and -2 which are the coefficient of x and y.
ii. Matrix of constant are 12 and -1.
iii. Matrix of unknown variables are x and y.

Summary This section discussed how linear equations will be solved using determinants and matrices.

SELF-ASSESSMENT QUESTIONS (SAQ 2)

1. Find the values of the unknown variables in the following system of linear equations using:
   a) Determinants
   b) Matrices

   □ $3x + 4y$

   $= 13\ 5x - 2$

   $y = 4$

2. Given the following system of linear equations

$$p + q + r = 7$$
$$2p - q + 3r = 4$$
$$3q - 6r = 0$$

   Find the values of p, q and r using
   a) Determinants
   b) Matrices

3. Given that:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 0 \\ 0 & 7 & -2 \end{pmatrix}; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; b = \begin{pmatrix} 15 \\ 10 \\ 8 \end{pmatrix}$$

   a) Formulate a system of linear equations to reflect the equation AX = b.
   b) Determine the values of the unknown variables using:
      i. Determinants
      ii. Matrices

STUDY SESSION 3: INPUT-OUTPUT ANALYSIS

Introduction

Quite often, there may exist some degree of interdependence among manufacturing organizations operating in the same industry. Such organizations may depend on one another for the supply of raw material inventory. This means that apart from satisfying inter-industry requirements, these organizations must also produce to satisfy final consumer demand. This interdependence is as a result of the fact that part of the output of an organization may be used as an input in the manufacture of another product by another organization. Thus, apart from satisfying its own requirements, an organization may have to give part of its own output as input for other organizations while the remaining output goes to the final consumers.

Input-output analysis, therefore, is that aspect of matrix algebra concerned with ascertaining industrial output levels so that both industrial and final consumer demands are met.

Objective

At the end of this session, the reader is expected to be able to use matrix algebra to solve input-output problems.

Main Text

Let us assume a hypothetical case of a 2-industry economy with industries A and B. Assume also that for A to produce 1 unit of output, it requires a_{11} of its own output and a_{21} A's output of B's output, and for B to produce 1 unit of output, it requires and a_{12} of A's output and a_{22} of its own output.

This information can be represented as follows:

	Industry A	Industry B
Industry A	a_{11}	a_{12}
Industry B	a_{21}	a_{22}

The above inter-industry requirements can be represented in the form of a TECHNOLOGICAL matrix, which is always denoted by a capital letter A as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Each row in matrix A represents the quantum of output (which may be expressed in Naira or units) expected from each industry, while column elements represent the quantum of input required from an industry.

Suppose the outputs of industries A and B are x_1 and x_2 , respective, then the output matrix X will be given by:

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Suppose also that final consumer demand for the output of industries A and B are d_1 and d_2, respectively, then, the demand matrix, D will be given by:

$$D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

The information contained in matrices A, X and D can be written as equations (1) and (2) as follows:

$$x_1 = a_{11} x_1 + a_{12} x_2 + d_1 \text{..........}(1)$$

$$x_2 = a_{21} x_1 + a_{22} x_2 + d_2 \text{........}(2)$$

The above equations can be interpreted to mean that the outputs of industries A and B should be equal to inter-industry requirements and final consumer demand. Rewriting equations (1) and (2), we have:

$$x_1 - a_{11} x_1 - a_{12} x_2 = d_1 \text{.............}(3)$$

$$-a_{21} x_1 + x_2 - a_{22} x_2 = d_2 \text{.............}(4)$$

Factorizing equations (3) and (4) we have:

$$(-a_{11}) x_1 - a_{12} x_2 = d_1 \text{.............}(5)$$

$$-a_{21} x_1 + (1 - a_{22}) x_2 = d_2 \text{.............}(6)$$

Rewriting equations (5) and (6) in the form of matrices, we have:

$$\begin{pmatrix} (1-a_{11}) & -a_{12} \\ -a_{21} & (1-a_{22}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \text{.....}(7)$$

Note that the first matrix in equation (7) can be obtained by subtracting the technological matrix (i.e. matrix A) from an identity matrix of the same size as matrix A.

That is:

$$\begin{pmatrix} (1-a_{11}) & -a_{21} \\ -a_{21} & (1-a_{22}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

In other words,

$$\begin{pmatrix} (1-a_{11}) & -a_{12} \\ -a_{21} & (1-a_{22}) \end{pmatrix} = I - A$$

Now, substituting (1 – A), X, and D into equation (7), we have:

(1 – A)X = D ....(8)

Note that the concern in input-output analysis is to determine the industrial output levels required to satisfy both inter-industry requirements as well as final consumer demand.

That is, according to the foregoing information, we want to ascertain the values of x_1 and x_2. This can be done by solving for X in equation (8). We therefore make X the subject of the formula in equation (8) as follows:

$$(I - A)X = D$$

Dividing both sides by (I – A), we have:

$$X = \left(\frac{D}{I - A}\right)$$

Or

$$X = (I - A)^{-1} D$$

Note that (I – A) is a matrix. Therefore, $(I – A)^{-1}$ is the inverse of matrix (I – A), which is given by:

$$(I - A)^{-1} = \frac{1}{|1 - A|} \times Adjo \text{ int } of\, (I - A)$$

Where $|I - A|$ = determinant of I-A.

Illustrative Example 1

Suppose there two industries in an economy, industries A and B. To produce 1 unit of A's output requires $\frac{1}{40}$ of its own output and $\frac{1}{3}$ of B's output. To produce 1 unit of B's output requires $\frac{2}{5}$ of A's output and $\frac{3}{5}$ of its own output. If final consumer demand for the output of A and B are 2400 and 3600 units respectively, determine the output levels of industries A and B that will meet both inter-industry requirements and final consumer demand.

Solution

The following matrices can be formulated from the given information:

$$A = \begin{pmatrix} \frac{1}{4} & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{5} \end{pmatrix}; \quad X = \begin{pmatrix} x \\ y \end{pmatrix}; \quad D = \begin{pmatrix} 2400 \\ 3600 \end{pmatrix}$$

Solution is given by:

$$X = (I - A)^{-1}D$$

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But $I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{2}{5} \\ -\frac{1}{3} & \frac{2}{5} \end{pmatrix}$

$$|I - A| = \begin{vmatrix} \frac{3}{4} & -\frac{2}{5} \\ -\frac{1}{3} & \frac{2}{5} \end{vmatrix} = \left(\frac{3}{4} \times \frac{2}{5}\right) - \left(-\frac{1}{3} \times -\frac{2}{5}\right) = \frac{6}{20} - \frac{2}{15} = \frac{1}{6}$$

$$(I - A) = \begin{pmatrix} \frac{3}{4} & -\frac{2}{5} \\ -\frac{1}{3} & \frac{2}{5} \end{pmatrix}$$

Cofactor of $(I - A) = \begin{pmatrix} \frac{2}{5} & -\frac{-1}{3} \\ \frac{2}{5} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{3}{4} \end{pmatrix}$

$\therefore Adjoint\ of\ (I - A) = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{4} \end{pmatrix}$

Hence, $(I - A)^{-1} = \frac{1}{\frac{1}{6}} \times \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{4} \end{pmatrix}$

$$6\begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{12}{5} & \frac{12}{5} \\ \frac{6}{3} & \frac{18}{4} \end{pmatrix}$$

Recall that X = $(I - A)^{-1}D$

That is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{12}{5} & \frac{12}{5} \\ \frac{6}{3} & \frac{18}{4} \end{pmatrix} \begin{pmatrix} 2400 \\ 3600 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{12}{5} & \frac{12}{5} \\ \frac{6}{1} & \frac{18}{4} \end{pmatrix} \begin{pmatrix} 2400 \\ 3600 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{12}{4} \times 2400 + \frac{12}{5} \times 3600 \\ \frac{6}{1} \times 2400 + \frac{18}{4} \times 3600 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{28800}{5} + \frac{43200}{5} \\ \frac{11400}{3} + \frac{61800}{4} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{22000}{5} \\ \frac{252000}{12} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14400 \\ 21000 \end{pmatrix}$$

This means that industry A should produce 14,400 units of output while industry B should produce 21,000 units of output.

Illustrative Example 2

A manufacturing company produces three products, A, B and C, each of which is partly reabsorbed in production. In other words, each product requires proportions of the output of the remaining two products as inputs. An analysis of the production of these products shows the following proportions; to produce 1 unit of product A, the company requires $\frac{2}{10}$ unit of product B and $\frac{1}{10}$ unit of product C. To produce 1 unit of product B, the company requires $\frac{3}{10}$ unit of product A and $\frac{5}{10}$ unit of product C. And to produce 1 unit of product C, the company requires $\frac{4}{10}$ unit of product A and $\frac{3}{10}$ unit of B.

The production targets are 150,000 units of A, 200,000 units of B and 100,000 units of C. These figures are the number of units of each of the three products which should reach the final consumers. Determine how many units of each to be produced.

Solution

$$A=\begin{pmatrix} 0 & \frac{3}{10} & \frac{4}{10} \\ \frac{2}{10} & 0 & \frac{3}{10} \\ \frac{1}{10} & \frac{5}{10} & 0 \end{pmatrix};\quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix};\quad D=\begin{pmatrix} 150000 \\ 200000 \\ 100000 \end{pmatrix}$$

$$I-A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 & \frac{3}{10} & \frac{4}{10} \\ \frac{2}{10} & 0 & \frac{3}{10} \\ \frac{1}{10} & \frac{5}{10} & 0 \end{pmatrix}=\begin{pmatrix} 1 & \frac{-3}{10} & \frac{-4}{10} \\ \frac{-2}{10} & 1 & \frac{-3}{10} \\ \frac{-1}{10} & \frac{-5}{10} & 1 \end{pmatrix}$$

$$|I-A|=\left|\begin{matrix} 1 & \frac{-1}{10} & \frac{-4}{10} \\ \frac{-2}{10} & 1 & \frac{-3}{10} \\ \frac{-1}{10} & \frac{-5}{10} & 1 \end{matrix}\right|\begin{matrix} 1 & \frac{-3}{10} \\ \frac{-2}{10} & 1 \\ \frac{-1}{10} & \frac{-5}{10} \end{matrix}$$

$$|I-A| = (1\times 1)+\left(\frac{-3}{10}\times\frac{-3}{10}\times\frac{-1}{10}\right)+\left(\frac{-4}{10}\times\frac{-2}{10}\times\frac{-5}{10}\right)-\left(\frac{-1}{10}\times 1\times\frac{-4}{10}\right)+\left(\frac{-5}{10}\times\frac{-3}{10}\times 1\right)+\left(1\times\frac{-2}{10}\times\frac{-3}{10}\right)$$

$$|I-A| = \left(1-\frac{9}{1000}-\frac{40}{1000}\right)-\left(\frac{4}{100}+\frac{15}{100}+\frac{6}{100}\right)$$

$$|I-A| = \frac{951}{1000}-\frac{25}{100}$$

$$|I-A| = \frac{701}{1000}$$

$$\text{Cofactor of } (I-A)=\begin{pmatrix} +\left|\begin{matrix} 1 & \frac{-3}{10} \\ \frac{-5}{10} & 1 \end{matrix}\right| & -\left|\begin{matrix} \frac{-2}{10} & \frac{-3}{10} \\ \frac{-1}{10} & 1 \end{matrix}\right| & +\left|\begin{matrix} \frac{-2}{10} & 1 \\ \frac{-1}{10} & \frac{-5}{10} \end{matrix}\right| \\ -\left|\begin{matrix} \frac{-3}{10} & \frac{-1}{10} \\ \frac{-5}{10} & 1 \end{matrix}\right| & +\left|\begin{matrix} 1 & \frac{-1}{10} \\ \frac{-1}{10} & 1 \end{matrix}\right| & -\left|\begin{matrix} 1 & \frac{-3}{10} \\ \frac{-1}{10} & \frac{-5}{10} \end{matrix}\right| \\ +\left|\begin{matrix} \frac{-3}{10} & \frac{-4}{10} \\ 1 & \frac{-3}{10} \end{matrix}\right| & -\left|\begin{matrix} 1 & \frac{-1}{10} \\ \frac{-2}{10} & \frac{-3}{10} \end{matrix}\right| & +\left|\begin{matrix} 1 & \frac{-3}{10} \\ \frac{-2}{10} & 1 \end{matrix}\right| \end{pmatrix}$$

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$$\text{Cofactor of }(I-A)=\begin{pmatrix} +\left(1-\frac{15}{100}\right) & -\left(\frac{2}{10}-\frac{-3}{100}\right) & +\left(\frac{10}{100}+\frac{1}{10}\right)\\ -\left(\frac{-3}{10}-\frac{20}{100}\right) & +\left(1-\frac{4}{100}\right) & -\left(\frac{-5}{10}-\frac{3}{100}\right) \\ +\left(\frac{9}{100}+\frac{4}{10}\right) & -\left(\frac{-3}{10}-\frac{8}{100}\right) & +\left(1-\frac{6}{100}\right)\end{pmatrix}$$

$$\text{Cofactor of }(I-A)=\begin{pmatrix} +\frac{85}{100} & -\frac{23}{100} & +\frac{20}{100}\\ -\frac{-50}{100} & +\frac{96}{100} & -\frac{-53}{100} \\ +\frac{49}{100} & -\frac{38}{100} & +\frac{94}{100}\end{pmatrix}$$

$$\text{Cofactor of }(I-A)=\begin{pmatrix} \frac{85}{100} & \frac{23}{100} & \frac{20}{100}\\ \frac{50}{100} & \frac{96}{100} & \frac{53}{100} \\ \frac{49}{100} & \frac{38}{100} & \frac{94}{100}\end{pmatrix}$$

$$\therefore Adjo\text{ int } of\ (I-A)=\begin{pmatrix} \frac{85}{100} & \frac{50}{100} & \frac{49}{100}\\ \frac{23}{100} & \frac{96}{100} & \frac{38}{100} \\ \frac{20}{100} & \frac{53}{100} & \frac{94}{100}\end{pmatrix}$$

$$\text{Hence, } ((I-A))^{-1}=\frac{1}{\frac{701}{100}}\times\begin{pmatrix} \frac{85}{100} & \frac{50}{100} & \frac{49}{100}\\ \frac{23}{100} & \frac{96}{100} & \frac{38}{100} \\ \frac{20}{100} & \frac{53}{100} & \frac{94}{100}\end{pmatrix}$$

$$(I-A)^{-1}=\frac{100}{701}\begin{pmatrix} \frac{85}{100} & \frac{50}{100} & \frac{49}{100}\\ \frac{23}{100} & \frac{96}{100} & \frac{38}{100} \\ \frac{20}{100} & \frac{53}{100} & \frac{94}{100}\end{pmatrix}$$

$$(I-A)^{-1}=\begin{pmatrix} \frac{8500}{70100} & \frac{5000}{70100} & \frac{4900}{70100}\\ \frac{2300}{70100} & \frac{9600}{70100} & \frac{3800}{70100} \\ \frac{2000}{70100} & \frac{5300}{70100} & \frac{9400}{70100}\end{pmatrix}$$

$$((I-A))^{-1}=\begin{pmatrix} \frac{85}{701} & \frac{50}{701} & \frac{49}{701}\\ \frac{23}{701} & \frac{96}{701} & \frac{38}{701} \\ \frac{20}{701} & \frac{53}{701} & \frac{94}{701}\end{pmatrix}$$

But X = $(I-A)^{-1}D$

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That is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{85}{701} & \frac{50}{701} & \frac{49}{701} \\ \frac{23}{701} & \frac{96}{701} & \frac{38}{701} \\ \frac{20}{701} & \frac{53}{701} & \frac{94}{701} \end{pmatrix} \begin{pmatrix} 150000 \\ 200000 \\ 100000 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} \frac{85}{701} \times 150000 + \frac{50}{701} \times 200000 + \frac{49}{701} \times 100000 \\ \frac{23}{701} \times 150000 + \frac{96}{701} \times 200000 + \frac{38}{701} \times 100000 \\ \frac{20}{701} \times 150000 + \frac{53}{701} \times 200000 + \frac{94}{701} \times 100000 \end{matrix}$$

$$\begin{pmatrix} x \\ \\ z \end{pmatrix} \begin{matrix} \frac{12750000}{701} + \frac{10000000}{701} + \frac{4900000}{701} \\ 701 + \frac{49200000}{701} + \frac{3800000}{701} \\ \frac{3000000}{701} + \frac{10600000}{701} + \frac{9400000}{701} \end{matrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{27650000}{701} \\ \frac{26450000}{701} \\ \frac{23000000}{701} \end{pmatrix} = \begin{pmatrix} 39{,}443.65 \\ 37{,}731.81 \\ 32{,}810.27 \end{pmatrix}$$

In-Text Questions (ITQ)

In-Text Answer (ITA)

Summary; this section discussed how manufacturing organizations that are in the same industry depend on each other for the supply of material inventory in order to satisfy the final consumer.

SELF-ASSESSMENT QUESTIONS (SAQ 3)

1. Given that

$$A = \begin{pmatrix} \frac{1}{4} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{pmatrix}; \; X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \; D = \begin{pmatrix} 860 \\ 920 \end{pmatrix}$$

Find the values of X_1 and X_2

2. To produce ₦1 of industry P’s output requires ₦0.30 of industry Q’s output. To produce ₦1 of industry Q’s output requires ₦0.20 of its own output and ₦0.10 of industry P’s output. How much of each

industry produce if non-industrial demand is ₦800 for industry P's output and ₦500 for industry Q's output?

References

Ho, S. T. *et al*. *College Mathematics, Vol.I*. Ibadan:African Educational Publishers. Karl, J.

S. (1990). *Mathematics for Business*, np: Win C. Brown Publishers.

Takyi-Asiedu, S. (1986). *Business Mathematics*, Bedfordshire: Graham Burn.

STUDY SESSION 4: BUSINESS APPLICATIONS OF DIFFERENTIAL CALCULUS

Introduction

Differential calculus has important areas of application in business. To a businessman, the overall goal is that of either the maximization of profit, sales volume or market share or the minimization of total or average cost. Differential calculus, which is also known as differentiation or mathematics of change, is a mathematical technique can be utilized to determine for instance, the points at which profit is highest or the points where the lowest cost occurs.

Areas of application of differential calculus in business include profit maximization and cost minimization, inventory control, breakeven output level etc.

Objectives

At the end of this session, the reader should be able to:

i) Apply differential calculus in solving problems involving profit maximization andcost minimization.
ii) Apply differentiation in solving inventory model problems.
iii) Apply differentiation to solve breakeven output level problems.

Main Text

a) Profit Maximization and Cost Minimization

Suppose Total Revenue, (TR), Total Cost (TC), and Profit, (P) are all functions of the number of units produced and sold, (q), then by definition, the Total Revenue function will be given by:

$$TR = r \times q = rq$$

Where r = revenue or price per unit

Q = number of units produced and sold

Similarly, the Total Cost function will be given by:

$$TC = FC + Vq$$

Where FC = Fixed Cost

Vq = Total Variable Cost

Profit, P is the difference between Total Revenue and Total Cost given by:

$$P = TR - TC$$

For maximum profit, we find the first derivative of the profit function and set it to zero, and since TR, TC, and P are all functions of the number of units, *q*, we differentiate *P* with respect to *q*.

That is:

$$P = TR - TC$$

$$\frac{dp}{dq} = \frac{d}{dq}(TR) - \frac{d}{dq}(TC)$$

For maximum profit, set $\frac{dp}{dq} = 0$, *i.e.*

$$\frac{d}{dq}(TR) - \frac{d}{dq}(TC) - 0$$

or

$$\frac{d}{dq}(TR) - \frac{d}{dq}(TC)$$

but $\frac{d}{dq}(TR) = MR$, marginal revenue, and $\frac{d}{dq}(TC) = MC$ = marginal cost

Thus, at maximum profit, MR = MC

Illustrative Example 1

The revenue earned by a company from each unit of its product sold is given by the function $r = 24$ $0.04q$. Variable cost per unit of q is given by $V = 640.02q$. If fixed cost is ₦1000:

i) Derive the Total Revenue and Total Cost functions.
ii) Determine the profit maximizing output level using both the profit function approach and the MR = MC approach.
iii) Determine the Total Cost at the profit maximizing output level.

Solution

i) Given that $r = 24 - 0.04q$. $V = 6 + 0.02q$ and $FC = ₦1000$.

$$TR = r \times q = rq$$
$$\square\ (24 - 0.04q)q$$
$$\square\ 24q - 0.04q^2$$

$$TC = FC + Vq$$

but $Vq = (6 + 0.02q)q$

$$Vq = 6 + 0.02q^{-2}$$
$$\therefore TC = 1000 + 6q + 0.02q^2$$

ii. a)The profit function approach

$$P = TR - TC$$
$$P = 24q - 0.04q^2 - (1000 + 6q + 0.02q^2)$$
$$P = 24q - 0.04q^2 - 1000 + 6q + 0.02q^2\ P$$
$$= 24q - 6q - 0.04q^2 - 0.02q^2 - 1000$$
$$P = 18q - 0.06q - 1000$$

$$\frac{dp}{dq} = 18 - 0.12q$$

For maximum profit, set $\frac{dp}{dq} - 0$, *i.e.*

18 – 0.12q = 0
-0.12q = -18
0.12q = 18
Q = 150 units

b) The MR = MC approach

$$MR = \frac{d}{dq}(TR)$$

$$\frac{d}{dq}(24q - 0.04q^2$$

$$24 - 0.08q$$

$$MC = \frac{d}{dq}(TC)$$

$$\frac{d}{dq}(1000 + 6q + 0.02q^2)$$

$$6 + 0.04q$$

At maximum profit, MR = MC, i.e.

24 – 0.08q = 6 + 0.04q
-0.08q – 0.04q = 6 – 24
-0.12q = -18
0.12q = 18
q = 150 units

TC, when q = 150
TC = 1000 + 6(150) + 0.02(150^2)
TC = 1000 + 6(150) + 0.02(22500)
TC = 1000 + 900 + 450
TC = ₦2,350

Illustrative Example 2

If profit, P = ₦4,800, FC = ₦18,000. Variable cost per unit = ₦8 and 1,900 units have been produced and sold:

i) What is the Total Revenue earned?
ii) At what selling price per unit was each sold?
iii) Derive the profit function for various levels of q
iv) What is the profit if q = 2,600 units?

Solution

Given: P = ₦4,800, FC = ₦18,000, V = ₦8, q = 1,900 units

i) P=TR–TC

4,800 = TR – (FC + Vq)

4,800 = TR – (18,000 + 8 x 1900)

4,800 = TR – (18,000 + 15,200)

4,800 = TR – 33,200

TR = 4,800 + 33,200

TR = ₦38,000

ii) Method 1.

SP per unit = $\frac{TR}{q} = \frac{38,000}{1,900} =$ ₦20

Method 2.

Let SP be x

$\therefore x \times q = TR$

$x \times 1,900 = TR$

1,900x = 38,000

$x = \frac{38,000}{1,900}$

x = N20

iii) P=TR–TC

P = 20q – (18,000 + 8q)

P = 20q – 18,000 – 8q

P= 12q – 18,000

iv) If q = 2,600

P = 12(2,600) – 18,000

P = 31,200 – 18,000

P = ₦13,200

In-Text Questions (ITQ)

What are the approaches used in solving profit maximization and cost minimization problems?

In-Text Answer (ITA)

i. The profit function approach which is P= TR – TC
ii. The marginal revenue marginal cost approach which is MR = MC

SELF-ASSESSMENT QUESTIONS (SAQ 4)

1. The price per unit of a firm's product is given by $r = 40 - 0.08x$. If variable cost per unit is given by $V = 12 - 0.01x$ and fixed cost is ₦8, find:
   a) the total revenue and total cost functions
   b) the marginal revenue and marginal cost functions

c) the profit maximizing output level using the MR = MC approach

d) the profit at an output level of 200 units.

2. A company's selling price per unit of its product is given by the function $r = 300 - 0.5x$ and its total cost function is given by $TC = 2000 + 100q$.

a) If the company's desire is to maximize profit, what number of units must it produce to achieve this objective?

b) What is the company's maximum profit?

c) What is the company's marginal cost function?

3. If revenue per unit is given by $r = 500 - 0.005x$, $FC = ₦48,000$, and variable cost per unit is given by $V = 100 + 0.003x$, find:

a) The Total revenue function

b) The Total Cost function

c) The profit function

d) The profit maximizing output level

e) The marginal revenue function

f) The maximum profit

b) Breakeven Output Level

A breakeven output situation is a zero profit situation where Total Revenue earned is just equal to Total Cost.

Suppose Total Cost, (TC), Marginal Cost, (MC) and Average Cost, (AC) are all functions of the number of units produced and sold, *q*.

$$MC = \frac{d}{dq}(TC), \text{ and } AC = \frac{TC}{q}$$

Now, differentiate AC with respect to q, i.e.;

$$\frac{d}{dq}(AC) = \frac{\frac{d}{dq}(TC) - TC \times 1}{q^2}$$

For minimum cost, set $\frac{d}{dq}(AC) = 0$

i.e.

$$\frac{\frac{d}{dq}(TC) - TC}{q^2} = 0$$

$$q\frac{d}{dq}(TC) - TC = 0 \times q^2$$

$$q\frac{d}{dq}(TC) - TC = 0$$

Divide both sides by *q*

i.e.

$$q \frac{d}{dq}(TC) = \frac{TC}{q}$$

But $\frac{d}{dq}(TC) = MC$, *and* $\frac{TC}{q} = AC$

Thus, $MC = AC$

That is, at the breakeven output level, $MC = AC$ and the selling price per unit is equal to MC (or AC).

Illustrative Example 1

A product's fixed cost is ₦1,250 and variable cost per unit is given by $6 + 0.02q$.

Determine:

i) The output level at which profit will be zero.

ii) The selling price per unit of the product.

Solution

Total variable cost, $Vq = (6 + 0.02q)q$

$$Vq = 6q + 0.02q^2$$

Total Cost, TC = $1{,}250 + 6q + 0.02q^2$

i). $MC = \frac{d}{dq}(TC)$

$$MC = \frac{d}{dq}(1{,}250 + 6q + 0.02q^2)$$

$$MC = 6 + 0.04q$$

$$AC = \frac{TC}{q}$$

$$AC = \frac{1{,}250 = 6q + 0.02q^2}{q}$$

At breakeven output level, MC = AC, i.e.

$$6 + 0.04q = \frac{1{,}250 + 6q + 0.02q^2}{q}$$

$$6q + 0.04q^2 = 1{,}250 + 6q + 0.02q^2$$

$$6q - 6q + 0.04q2 - 0.02q^2 = 1{,}250$$

$$q^2 = 62{,}500$$

$$q = \sqrt{62{,}500}$$

$q = 250$ units

Thus, the zero-profit output level is 250 units.

ii) Recall that at breakeven output level, MC = AC = Selling Price per unit.

$\therefore SP\ perunit = MC$

$\quad 6 + 0.04(250)$

$\quad 6+10$

$= ₦16$

Illustrative Example 2

A firm's fixed cost with respect to product *x* is ~~₦1~~35,000. Variable cost per unit of *x* is given by the function:

$$V = 300 + 0.015q$$

Determine:

i) The Total Cost function
ii) The Marginal and Average Cost functions
iii) The output level that equates profit to zero
iv) The selling price per unit of *x*

Solution

Total Variable cost, $Vq = (300 + 0.015q)q$

$$Vq = 300q + 0.015q^2$$

i) Total Cost, $TC = FC + Vq$

$$TC = 135{,}000 + 300qq + 0.015q^2$$

ii)

$$MC = \frac{d}{dq}(TC)$$

$$MC = \frac{d}{dq}(135{,}000 + 300q + 0.015q^2$$

$$MC = 300 + 0.03q$$

$$AC = \frac{TC}{q}$$

$$AC = \frac{135{,}000 + 300q + 0.015q^2}{q}$$

iii) At breakeven output level, MC = AC, i.e.

$300 + 0.03q = \frac{135{,}000 + 300q + 0.015q^2}{q}$

$300q + 0.03q^2 = 135{,}000 + 0.015q^2$

$300q - 300q + 0.03q - 0.015q^2 = 135{,}000$

$0.015q^2 = 135{,}000$

$q^2 = 9{,}000{,}000$

The breakeven output level is 3,000 units.

iv) At breakeven output level, SP per unit is equal to MC (or AC), i.e.

SP per unit $= 300 + 0.03q$

The breakeven output level is 3000 units.

$\therefore SP\ per\ unit = 300 + 0.03(3000)$

$= 300 + 90$

$=$ ₦390

In- Text Questions (ITQ)

What is breakeven output level?

In-Text Answer (ITA)

The breakeven output level it is a level where profit is at zero which means total revenue is equal to total cost TR = TC

SELF-ASSESSMENT QUESTIONS (SAQ 5)

1. If revenue per unit is given by $r = 500 - 0.005x$, $FC =$ ₦*48,000*, and variable cost per unit is given by $V = 100 + 0.003x$, find:
   a) The average cost function
   b) The marginal cost function
   c) The output level that equates total revenue to total cost
   d) The selling price per unit at the breakeven output level

2) A company's fixed cost in respect of a product is ₦15,000. Variable cost per unit is given by $V = 2 + 0.08x$. Determine:
   a) The marginal and average cost functions
   b) The breakeven output level
   c) The product's selling price per unit at the breakeven output level

Inventory Control

Inventory control is another area of interest to the businessman. An important aspect of inventory that concerns the businessman is its associated costs. Inventories can take the form of raw materials, work-in-progress or finished goods, all of which have cost implications for the business. For instance, keeping high levels of raw materials inventory can guaranty long and uninterrupted production runs and prevents customer dissatisfaction or loss of goodwill due to stock outs and thereby enhances sales, but then the businessman must have to cope with the high inventory holding cost associated with inventory levels. On the other hand, if the businessman chooses to keep low inventory levels in order to cut down on inventory holding costs, he would have to cope with the economic consequences of running out of stock (e.g. costs of customer dissatisfaction and goodwill loss) as well as high ordering cost as a result of the frequency with which orders are placed for inventories. The businessman is therefore at the centre of two inventory management problems – overstocking on the one side and under stocking on the other.

The Economic Order Quantity (EOQ) Model

The EOQ model is a quantitative technique designed to maintain a system of inventory control which minimizes both inventory ordering and holding costs (i.e. Total Cost) by establishing the optimum level of stock to be ordered.

Assumptions of EOQ Model

The EOQ model is based on the assumptions that:

- i) Demand rate is constant, recurring and known with certainty;
- ii) Lead time, i.e. the time between order placement an order delivery, is known and does not change;
- iii) Discounts are not allowed;
- iv) Ordering and inventory holding costs are known.

Suppose:
R = annual requirement or demand
Q = quantity ordered per order
C_o = ordering cost per order
C_H = inventory holding cost per unit per year

Then:

Average inventory = $\frac{Q}{2}$

Number of orders per year = $\frac{annual requirement}{quantity ordered per order}$

$$= \frac{R}{Q}$$

$$\therefore Total\ ordering\ \cos t = \frac{R}{Q} \times C_o$$

$$= \frac{R}{Q} C_o$$

Total inventory holding cost = $\frac{R}{Q} \times C_H$

$$= \frac{R}{Q} C_H$$

$$\therefore Total\ Cost,\ TC = \frac{R}{Q} C_o + \frac{Q}{2} C_H$$

$$\frac{dTC}{dQ} = RC_o Q^{-2} + \frac{C_H}{2}$$

$$= \frac{-RC_o}{Q^2} + \frac{C_H}{2}$$

To find the Q that minimizes total cost, we set $\frac{dTC}{dQ} = 0$

i.e. $\frac{-RC_o}{Q^2} + \frac{C_H}{2} = 0$

or $\frac{C_H}{2} = \frac{RC_o}{Q^2}$

By cross-multiplication, we have:

$$Q^2 C_H = 2RC_o$$

$$Q^2 = \frac{2RC_o}{C_H}$$

$$Q = \sqrt{\frac{2RC_o}{C_H}}$$

Illustrative Example 1

A company's annual requirement for raw materials is 500,000 tones. If ordering cost per order is ₦20 and inventory holding cost per unit per year is ₦0.20 and that the company operates for 280 days in a year with a lead-time of 8 days, find:

i) The optimum order quantity
ii) The number of orders to be placed for raw materials annually
iii) The total cost at the optimum order quantity
iv) The cycle time
v) The reorder point

Solution

i) $Q = \sqrt{\frac{2 \times 500{,}000 \times 20}{0.20}}$

$= \sqrt{\frac{20{,}000{,}000}{0.20}}$

$= \sqrt{100{,}000{,}000}$

= 10,000 units

ii) Number of orders = $\frac{\text{Annual requirements}}{EOQ}$

$= \frac{500{,}000}{10{,}000}$

= 50 orders

iii) Total Cost, TC = $\frac{R}{Q} \times C_o + \frac{Q}{2} \times C_H$

$= \frac{500,000}{10,000} \times 20 + \frac{10,000}{2} \times 0.20$

= 1,000 + 1,000

= ₦2,000

iv) Cycle Time = $\frac{Q}{R}$ × Number of working days in a year

$= \frac{10,000}{500,000} \times 280$

= 0.02 x 280

= 50 days

v) Reorder point = Average daily usage x lead time

$= \frac{500,000}{280} \times 7$

= 12,500 units.

This means a new order for raw materials should be placed at this level of raw material inventory.

The Price-Break Model

One of the assumptions of the EOQ model is that discounts are not allowed. In reality however, sometimes businesses are transacted with discounts allowed to encourage prompt settlement of bills or to stimulate sales. This realization is incorporated into the price-break model (i.e. EOQ with discounts). The only difference in assumptions between the EOQ and the price-break model is that discounts are allowed in the latter. Under this model, the Total Cost function is given by:

$$TC = \frac{R}{Q} C_o + \frac{Q}{2} C_H + PR$$

Where P = Price per unit

R = Annual requirement

Is solving the price-break model, the following steps are followed:

i) Compute the EOQ for each quantity range. If the computed EOQ falls within the range, it is a feasible EOQ. Otherwise, it is not.

ii) Substitute the feasible EOQ in the TC function to determine total cost.

iii) For non-feasible EOQ, the lower limit of the quantity range is used to determine total cost.

iv) Select the lowest total cost. This corresponds to the best EOQ.

Illustrative Example 2

A firm obtained quantity discounts on its order of raw materials as follows:

Quantity range	**Price per unit**
1 and less than 100	₦1,000
100 and less than 200	₦950
200 and less than 500	₦800
500 and less than 800	₦750
800 and above	₦700

If annual requirement for materials is 4,000 tonnes, C_o is ₦500, C_H is 20% of price per unit, what is the optimum order quantity?

Solution

<u>1 and less than 100</u>

$$EOQ = \sqrt{\frac{2 \times 4{,}000 \times 500}{0.20 \times 1{,}000}}$$

$$= \sqrt{\frac{4{,}000{,}000}{200}}$$

$$= \sqrt{20{,}000}$$

= 141.42 units. This is a non-feasible EOQ.

$$TC = \frac{R}{Q} C_o + \frac{Q}{2} C_H + PR$$

$$TC = \frac{4{,}000}{1} \times \frac{1}{2} \times 200 + 1{,}000 \times 4{,}000$$

= 2,000,000 + 100 + 1,000 x 4,000

= ₦6,000,100

<u>100 and less than 200</u>

$$EOQ = \sqrt{\frac{2 \times 4{,}000 \times 500}{0.20 \times 950}}$$

$$= \sqrt{\frac{4{,}000{,}000}{190}}$$

$\sqrt{21,052.63}$

= 145 units. This is feasible EOQ.

$TC = \frac{4,000}{145} \times 500 + \frac{145}{2} \times 190 + 950 \times 4,000$

= 13,793.10 + 13,775 + 3,800,000
= ₦3,827,568.10

<u>200 and less than 500</u>

$EOQ = \sqrt{\frac{2 \times 4,000 \times 500}{0.20 \times 800}}$

$\sqrt{\frac{4,000,000}{160}}$

$\sqrt{25,000}$

= 158.11 units. This is non-feasible

$TC = \frac{4,000}{200} \times 500 + \frac{200}{2} \times 160 + 800 \times 4,000$

= ₦3,226,000

<u>500 and less than 800</u>

$EOQ = \sqrt{\frac{2 \times 4,000 \times 500}{0.20 \times 750}}$

$\sqrt{\frac{4,000,000}{150}}$

$\sqrt{26,666.67}$

= 163 units. This is non-feasible

$TC = \frac{4,000}{500} \times 500 + \frac{500}{2} \times 150 + 750 \times 4,000$

= 4,000 + 37,500 + 3,000,000

= ₦3,041,500

<u>800 and above</u>

$$EOQ = \sqrt{\frac{2 \times 4,000 \times 500}{0.20\ 700}}$$

$$\sqrt{\frac{4,000,000}{140}}$$

$$\sqrt{28,571.43}$$

169 units. This is non-feasible

$$TC = \frac{4,000}{800} \times 500 + \frac{800}{2} \times 140 + 700 \times 4,000$$

= 2,500 + 56,000 + 2,800,000

= ₦2,858,500

Since the lowest cost is ₦2,858,500, the corresponding EOQ of 169 units is the optimum order quantity.