Rotation and General Plane Motion 15

Learning objectives : The key concepts included in this chapter relate to the following aspects of rotation and general plane motion

- *angular displacement, velocity and acceleration; relationship between circular motion and linear motion*
- *equations of motion is a circular path with uniform angular acceleration*
- *general plane motion and Chasle's theorem*
- *instantaneous centre and its applications*
- *Coriolis' law*

The motion of a particle in curvilinear motion is on a curved path. The particle does not repeat the journey once it has passed over a point on the curved path. Its centre of rotation also goes on changing.

In the rotary (circular) motion, the movement of the particle is along a circular path. The particle repeats its journey along the same circular path about the same centre of rotation which remains fixed. The shafts, pulleys and flywheels etc., undergo circular motion when they rotate about their geometric axis.

15.1 TERMS RELATED TO CIRCULAR MOTION

The terms associated with circular motion are:

(i) Angular displacement: The displacement of a body in rotation is called angular displacement, and it is measured in terms of the angle through which the body moves from the initial state. With reference to Fig. 15.1, a body starts from initial position A and while moving in circular path reaches the position B after a certain time interval. Then $\angle AOB = \theta$ is a measure of the angular displacement of the body. This displacement has magnitude as well as direction, and apparently displacement is a vector quantity. The direction is a rotation-either clockwise or anti-clockwise.

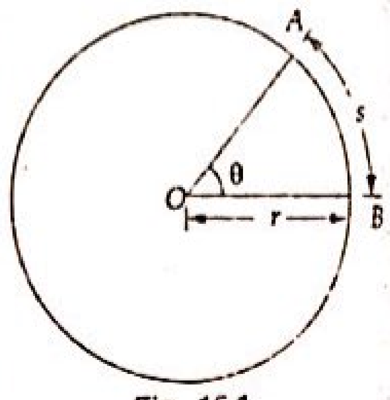


Fig. 15.1

Angular displacement θ can be measured in radians, degrees or revolutions. Various units of angular displacement are connected by the relation:

$$1 \text{ revolution} = 2\pi \text{ radians} = 360 \text{ degrees}$$

(ii) Angular velocity: The rate of change of angular displacement of a body with respect to time is called angular velocity.

If the body traverses angular distance $d\theta$ over a time interval dt, then the average angular velocity ω is

$$\omega = \frac{d\theta}{dt}$$

The unit of angular velocity for defining its magnitude is usually taken as radians per second (rad/s). Many a times angular velocity is expressed in terms of revolutions per minute (rpm). If the body turns N rev/min, then

$$\text{number of revolutions per sec} = \frac{N}{60}$$

In one revolution, the angular displacement of the body is 360 degrees or 2π radians

Angle traversed by the body is one second

= (angle covered in one revolution) × (number of revolutions per second)

$$= 2\pi \times \frac{N}{60} = \frac{2\pi N}{60}$$

But the angle traversed by the body in one second is a measure of the angular velocity w. Then

$$\omega = \frac{2\pi N}{60} \text{ rad/sec}$$

(*iii*) *Angular acceleration:* The rate of change of angular velocity of a body with respect to time is called angular acceleration.

angular acceleration = rate of change of angular velocity

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{\theta}{t}\right) = \frac{d^2\theta}{dt^2}$$

The angular acceleration is sometimes expressed as

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} \qquad \left(\because \frac{d\theta}{dt} = \omega\right)$$

The unit of angular acceleration for defining its magnitude is usually taken as radian per second per second (rad/sec^2)

Relationship between circular motion and linear motion: When a body moves from A to B (Fig. 15.1), the distance travelled by it is s. If r is the distance of the body from the centre of rotation, then

$$s = r\theta$$

and

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} \qquad (\because r \text{ is constant})$$

Noting that ds/dt = linear velocity v and $d\theta/dt$ = angular velocity ω, we may write

$$v = \omega r$$

That is, the linear velocity is r-times the angular velocity

When the expression $v = \omega r$ is differentiated with respect to time, we get

$$\frac{dv}{dt} = \frac{d}{dt}(\omega r) = r\frac{d\omega}{dt}$$

Since dv/dt = linear acceleration a and $d\omega/dt$ = angular acceleration α, we may write

$$a = \alpha r$$

That is, the linear acceleration is r-times the angular acceleration.

It is worthwhile to point out that when a body moves along a curved path (curvilinear motion), the linear velocity is referred to as tangential velocity. Further, the body undergoes tangential and normal components of acceleration given by

tangential acceleration $a_t = \alpha r$

normal acceleration $a_n = \frac{v^2}{r} = \omega^2 r$

The tangential acceleration is due to change in magnitude of the velocity and the normal acceleration is due to change in direction of the body.

Fig. 15.2 illustrates these components for any point P. The normal component is directed towards the centre of the circle. The total acceleration a is the vector sum of the two components.

$$a = \sqrt{\alpha_n^2 + \alpha_t^2}$$

and the angle between the total acceleration and the radius is

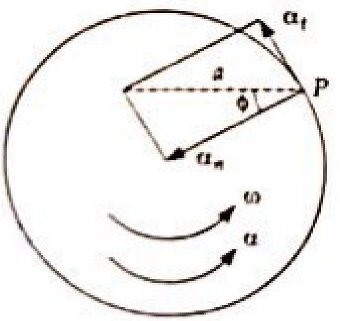


Fig. 15.2

$$\phi = \tan^{-1}\left(\frac{\alpha_t}{\alpha_n}\right)$$

15.2. EQUATIONS OF CIRCULAR MOTION

When a body moves in a circular path with uniform angular acceleration, the equations of motion are:

$$\omega = \omega_0 + \alpha t; \quad \omega^2 - \omega_0^2 = 2\,\alpha\theta; \quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

where ω_0 is the initial angular velocity, ω is the final angular velocity, α is angular acceleration of the body and θ represents the angle moved in time t.

(*i*) angular acceleration α = rate of change of angular velocity

$$= \frac{\text{change of angular velocity}}{\text{time}} = \frac{\omega - \omega_0}{t}$$

$$\omega - \omega_0 = \alpha t$$

$$\therefore \quad \omega = \omega_0 + \alpha t$$

Alternatively:

$$\alpha = \frac{d\omega}{dt} \;; \quad d\omega = \alpha\, dt$$

Integrating both sides within the limit $\omega = \omega_0$ at $t = 0$ and $\omega = \omega$ at $t = t$

$$\int_{\omega_0}^{\omega} d\omega = \alpha \int_0^t dt \qquad (\because \alpha \text{ is constant})$$

$$\omega - \omega_0 = \alpha t$$

$$\therefore \quad \omega = \omega_0 + \alpha t \qquad \text{...(15.1)}$$

(*ii*) Angle travelled = average angular velocity × time

$$\theta = \frac{\omega_0 + \omega}{2} \times t = \frac{\omega_0 + \omega_0 + \alpha t}{2} \times t \qquad (\because \omega = \omega_0 + \alpha t)$$

$$= \left(\omega_0 + \frac{\alpha t}{2}\right) t = \omega_0 t + \frac{1}{2}\alpha t^2$$

Alternatively:

$$\omega = \frac{d\theta}{dt} \quad ; \quad d\theta = \omega\, dt = (\omega_0 + \alpha t)\, dt$$

Upon integration

$$\int_0^\theta d\theta = \int_0^t (\omega_0 + \alpha t) dt$$

assuming $\theta = 0$ at $t = 0$ and $\theta = \theta$ at $t = t$

$$\theta = \left| \omega_0 t + \frac{\alpha t^2}{2} \right|_0^t = \omega_0 t + \frac{1}{2} \alpha t^2 \qquad ...(15.2)$$

(*iii*) Angle moved = average angular velocity × time

$$\theta = \frac{\omega_0 + \omega}{2} \times \frac{\omega - \omega_0}{\alpha} = \frac{\omega^2 - \omega_0^2}{2\alpha} \qquad \left(\because \omega = \omega_0 + \alpha t, \quad t = \frac{\omega - \omega_0}{\alpha} \right)$$

$$\therefore \quad \omega^2 - \omega_0^2 = 2\alpha\theta$$

Alternatively:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \frac{d\omega}{d\theta} \times \omega$$

or $\quad \alpha\, d\theta = \omega\, d\omega$

upon integration and assuming $\omega = \omega_0$ at $\theta = 0$ and $\omega = \omega$ at $\theta = \theta$

$$\alpha \int_0^\theta d\theta = \int_{\omega_0}^{\omega} \omega\, d\omega$$

$$\alpha\,\theta = \left| \frac{\omega^2}{2} \right|_{\omega_0}^{\omega} ; \quad \alpha\theta = \frac{\omega^2 - \omega_0^2}{2}$$

$$\therefore \quad \omega^2 - \omega_0^2 = 2\alpha\theta \qquad ...(15.3)$$

EXAMPLE 15.1.

A flywheel rotates freely on friction less bearings at 240 rev/min. How many revolutions will it make in 10 seconds after the start? Proceed to determine the angular speed if this wheel turns 500 revolutions in 15 seconds.

Solution: There will be no resistance to motion in frictionless bearings. Obviously the wheel will then rotate with uniform motion for which $\theta = \omega t$

(*i*) $\quad \omega = 240$ rev/min; $t = 10$ seconds

$$\therefore \quad \theta = \omega t = 240 \times \frac{10}{60} = 40 \text{ revolutions} = 40 \times 2\pi = 80\,\pi \text{ radians}$$

(*ii*) $\quad \theta = 500$ revolutions ; $t = 15$ seconds

$$\therefore \quad \omega = \frac{\theta}{t} = \frac{500}{(15/60)} = 2000 \text{ rev/min} = 2000 \times \frac{2\pi}{60} = 209.33 \text{ rad/s}$$

EXAMPLE 15.2.

The speed of a flywheel changes from 10 rad/s to 30 rad/s in 5 seconds time. Determine the angular acceleration of the wheel.

How many revolutions the wheel would turn to attain a speed of 600 rev/min?

Solution: Using the kinematic relation $\omega = \omega_0 + \alpha t$, we have

$$\text{angular acceleration } \alpha = \frac{\omega - \omega_0}{t} = \frac{30 - 10}{5} = 4 \text{ rad/s}^2$$

(b) $$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 600}{60} = 62.8 \text{ rad/s}$$

Then from the relation, $\omega^2 - \omega_0^2 = 2\alpha\theta$,

$$\text{total angular displacement } \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{62.8^2 - 10^2}{2 \times 4} = 480.48 \text{ radians.}$$

Now, 1 revolution ≡ 2π radians

∴ $$\text{Total revolutions made by the wheel} = \frac{480.48}{2\pi} = 76$$

EXAMPLE 15.3.

A roller of a motorised mortar mixer rotates for 5 seconds with a uniform angular acceleration and describes 120 radians during this time. It then rotates with constant angular velocity and covers 100 radians during the next 5 seconds. Make calculations for the initial angular velocity and the angular acceleration.

Solution: Let ω be the constant angular velocity during the later 5 seconds period of motion. Then

$$\omega = \frac{\text{angular displacement}}{\text{time}} = \frac{100}{5} = 20 \text{ rad/s}$$

If ω_0 is the initial angular velocity and α is the constant angular acceleration during the first 5 seconds of motion, then from the kinematic relations

$$\omega = \omega_0 + \alpha t \text{ and } \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

we have $$20 = \omega_0 + 5\alpha \quad \text{and} \quad 120 = 5\omega_0 + \frac{1}{2} \times \alpha \times 25$$

From these two expressions, we get

$$120 = 5(20 - 5\alpha) + \frac{1}{2}\alpha \times 25 = 100 - 25\alpha + \frac{25\alpha}{2}$$

or $$\frac{-25\alpha}{2} = 20$$

∴ $$\alpha = -\frac{2 \times 20}{25} = -1.6 \text{ rad/s}^2$$

and $$\omega_0 = 20 - 5\alpha = 20 - 5(-1.6) = 28 \text{ rad/s}$$

EXAMPLE 15.4.

A grinding wheel is attached to the shaft of an electric motor of rated speed 1800 rev/min. When the power is switched on, the unit attains the rated speed in 5 seconds. Assuming uniformly accelerated motion, determine the number of revolutions the wheel turns to attain the rated speed.

Solution: $\omega_0 = 0$; $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1800}{60} = 188.4$ rad/s

From the kinematic relation: $\omega = \omega_0 + \alpha t$, we have

$$\text{angular acceleration } \alpha = \frac{\omega - \omega_0}{t} = \frac{188.4 - 0}{5} = 37.68 \text{ rad/s}^2$$

Now from the relation: $\omega_2 - \omega_0^2 = 2\alpha\theta$

$$\text{total angular displacement } \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{188.4^2 - 0}{2 \times 37.68} = 471 \text{ radians}$$

Since, 1 revolution ≡ 2π radians

$$\text{total revolutions turned by the wheel} = \frac{471}{2\pi} = 75$$

EXAMPLE 15.5.

A 5 m long slender bar is made to rotate through its one end in the horizontal plane about a vertical axis. The bar accelerates uniformly from 1000 rev/min to 1500 rev/min in 5 second period. Determine:

(a) linear velocity of the mid-point of the bar at the beginning and at the end of the time interval.

(b) normal and tangential accelerations of the mid point of the bar 3 seconds after the start of acceleration.

Solution: The mid-point *M* of the slender bar describes a circular path and its linear velocity is given by: $v = wr$. Let the subscripts *b* and *e* denote the beginning and end of the time interval.

At the start of 5s interval,

$$v_b = \frac{2\pi \times 1000}{60} \times 3 = 314 \text{ m/s} \qquad \left(\because \omega = \frac{2\pi N}{60}\right)$$

At the end of 5s interval,

$$v_e = \frac{2\pi \times 1500}{60} \times 3 = 417 \text{ m/s}$$

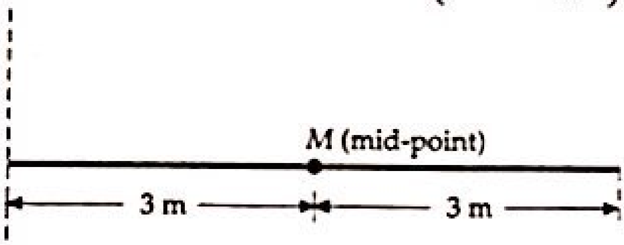


Fig. 15.3

Angular acceleration $\alpha = \frac{\omega - \omega_o}{t}$

$$= \frac{1}{5}\left[\frac{2\pi \times 1500}{60} - \frac{2\pi \times 1000}{60}\right] = 10.47 \text{ rad/s}^2$$

After 3 seconds, $\omega = \omega_o + \alpha t = \frac{2\pi \times 1000}{60} + 10.47 \times 3$

$$= 136.08 \text{ rad/s}$$

The tangential and normal components of acceleration are:

$$\alpha_t = \alpha r = 10.47 \times 3 = 31.41 \text{ m/s}^2$$

$$\alpha_n = \omega^2 r = (136.08)^2 \times 3 = 55553.3 \text{ m/s}^2$$

EXAMPLE 15.6.

The piston of an engine is stated to make 360 strokes per minute. Determine the angular and linear speed of a point on the periphery of 1.8 m diameter flywheel fitted to the crank shaft of the engine.

Solution: Corresponding to two strokes of the piston, there is one revolution of the crank shaft.

∴ Number of revolutions made by the crank shaft $= \frac{360}{2} = 180$ rev/min

Since the flywheel is rigidly fixed to the crankshaft.

Number of revolutions made by the flywheel = 180 rev/min

∴ Angular speed of the flywheel $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 180}{60} = 18.84$ rad/s

The linear velocity and the angular velocity are connected by the relation:

$$v = \omega r$$

∴ Linear speed $v = 18.84 \times \left(\frac{1.8}{2}\right) = 16.96$ m/s

EXAMPLE 15.7.

A flywheel 0.5 m in diameter accelerates uniformly from rest to 360 rpm in 12 seconds. Determine the velocity and acceleration of a point on the rim of the flywheel 0.1 second after it has started from rest.

Solution: $\omega_0 = 0$; $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 360}{60} = 37.68$ rad/s

From the kinematic relation: $\omega = \omega_0 + \alpha t$, we have

$$\text{angular acceleration } \alpha = \frac{\omega - \omega_0}{t} = \frac{37.68 - 0}{12} = 3.14 \text{ rad/s}$$

Angular velocity of the wheel 1 second after starting

$$\omega = \omega_0 + \alpha t = 0 + 3.14 \times 1 = 3.14 \text{ rad/s}$$

The linear velocity and the angular velocity are connected by the relation:

$$v = wr$$

∴ linear speed $v = 3.14 \times \frac{0.5}{2} = 0.785$ m/s

The linear velocity will be tangential to the rim.

The components of acceleration are:

$$\text{tangential acceleration } \alpha_t = \alpha r = 3.14 \times \frac{0.5}{2} = 0.785 \text{ m/s}^2$$

$$\text{normal acceleration } \alpha_n = \omega^2 r = (3.14)^2 \times \frac{0.5}{2} = 1.232 \text{ m/s}^2$$

Magnitude of total acceleration,

$$\alpha = \sqrt{\alpha_t^2 + \alpha_n^2} = \sqrt{0.785^2 + 1.232^2} = 1.461 \text{ m/s}^2$$

Let ϕ be the angle between the total acceleration vector and the radius

$$\phi = \tan^{-1}\frac{\alpha_t}{\alpha_n} = \tan^{-1}\left(\frac{0.785}{1.232}\right) = 32.5°$$

EXAMPLE 15.8

A shaft supported between two bearings carries a wheel of 1.5 m diameter. A constant moment is applied at the rim of the wheel and it attains a speed of 120 revolutions per minute in 10 minutes from rest. Determine:

(a) the peripheral velocity of the wheel,

(b) the angular acceleration of the wheel, and

(c) the number of revolutions made by the wheel

Solution: Angular velocity, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 120}{60} = 12.56$ rad/s

$$\text{Radius of the wheel, } r = \frac{1.5}{2} = 0.75 \text{ m}$$

The angular and peripheral velocities are connected by the identity: $v = \omega r$

$\therefore$ peripheral speed of the wheel $v = 12.56 \times 0.75 = 9.42$ m/s

(b) From the kinematic relation $\omega = \omega_0 + \alpha t$, we have

$$\text{angular acceleration } \alpha = \frac{\omega - \omega_0}{t} = \frac{12.56 - 0}{10 \times 60} = 0.0209 \text{ rad/s}^2$$

(c) From the kinematic relation $\omega^2 - \omega_0^2 = 2\alpha\theta$

$$\text{angular displacement } \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{12.56^2 - 0^2}{2 \times 0.0209} = 3774 \text{ radians}$$

Since 2π radians = 1 revolution

$$\text{Total number of revolutions make by the wheel} = \frac{3774}{2\pi} = 601$$

EXAMPLE 15.9.

The rotor of an electric motor accelerates uniformly to its rated speed of 3600 rev/min from rest in 15 seconds. Subsequently the power is turned off and the rotor decelerates to stop. If a total time of 75 seconds elapses from start to stop, determine the revolutions turned during acceleration and deceleration. Comment on the greater time taken during stopping.

Solution: During acceleration phase

$$\omega_0 = 0 \quad \text{and} \quad \omega = \frac{2\pi N}{60}\frac{2\pi \times 3600}{60} = 376.8 \text{ rad/s}$$

From the relation $\omega = \omega_0 + \alpha t$, we have

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{376.80}{15} = 25.12 \text{ rad/s}$$

Angle θ covered in 15 seconds,

$$\theta = \omega_0 + \frac{1}{2}\alpha t^2$$

$$= 0 + \frac{1}{2} \times 25.12 \times 15^2 = 2826 \text{ rad}$$

$\therefore$ Number of revolutions taken to attain the rated speed,

$$n_{accn} = \frac{\theta}{2\pi} = \frac{2826}{2\pi} = 450 \text{ revolutions}$$

(b) During deceleration phase,

$$\omega_0 = 376.8 \text{ rad/s}; \quad \omega = 0 \quad \text{and} \quad t = 75 - 15 = 60s$$

Again from the relation $\omega = \omega_0 + \alpha t$, we have

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 376.8}{60} = -6.28 \text{ rad/s} \quad \text{(retardation)}$$

Angle turned during deceleration can be worked out from the relation

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

$$\text{That gives: } \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{0 - 376.8^2}{2 \times (-6.28)} = 11304 \text{ rad}$$

The corresponding revolutions taken by the rotor to come to stop

$$n_{retd} = \frac{11304}{2\pi} = 1800 \text{ revolutions}$$

The taking of greater time during stopping maybe attributed to the following facts:

- the starting velocity is more in the second case than in the first case
- the rotation of the rotor continues during stopping phase due to inertia of the rotor.

EXAMPLE 15.10.

A wheel accelerates uniformly from rest to a speed of 180 rpm in 0.5 seconds. It then rotates at that speed for 2 seconds before decelerating to rest in 0.3 seconds. Workout the revolutions made by the wheel during the entire time interval.

Solution: (*i*) During uniform acceleration (from $t = 0$ to $t = 0.5$ sec)

$$\omega_1 = \frac{2\pi \times 180}{60} = 18.84 \text{ rad/s}$$

$$\alpha_1 = \frac{\omega_1 - \omega}{t} = \frac{18.84 - 0}{0.5} = 37.68 \text{ rad/s}^2$$

From the kinematic relation $\omega^2 - \omega_0^2 = 2\alpha\theta$, we have

$$\theta_1 = \frac{\omega_1^2 - \omega_0^2}{2\alpha_1} = \frac{18.84^2 - 0^2}{2 \times 37.68} = 4.7 \text{ radians}$$

(*ii*) During uniform speed (from $t = 0.5$ sec to 2.5 sec)

$$\omega_2 = 18.84 \text{ rad/s}$$

$$\theta_2 = \omega_2 t = 18.84 \times 2 = 37.68 \text{ radians.}$$

(*iii*) During uniform deceleration (from $t = 2.5$ sec to 2.8 sec)

$$\omega_2 = 18.84 \text{ rad/s} \quad ; \quad \omega_3 = 0$$

$$\alpha_3 = \frac{\omega_3 - \omega_2}{t} = \frac{0 - 18.84}{0.3} = -62.8 \text{ rad/s}$$

From the kinematic relation $\omega^2 - \omega_0^2 = 2\alpha\theta$, we have

$$\theta_3 = \frac{\omega_3^2 - \omega_2^2}{2\alpha} = \frac{0^2 - 18.84^2}{2(-62.8)} = 2.83 \text{ radians}$$

$\therefore$ Total angular displacement $= \theta_1 + \theta_2 + \theta_3$

$= 4.7 + 37.68 + 2.83 = 45.22$ radians

Since 1 revolution $= 2\pi$ radians

Total revolutions turned by the wheel $= \frac{45.22}{2\pi} = 7.2$ revolutions

Alternatively: Refer Fig. 15.4 which represents the velocity-time graph for the wheel. The area under the velocity-time graph gives the angle turned during the entire time-interval.

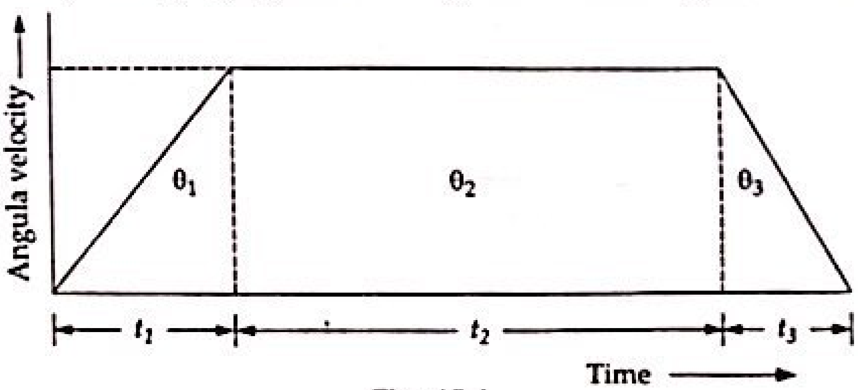


Fig. 15.4

$$\theta = \theta_1 + \theta_2 + \theta_3 = \left(\frac{1}{2} \times 0.5 \times 18.84\right) + (2 \times 18.84) + \left(\frac{1}{2} \times 0.3 \times 18.84\right)$$

$$= 4.71 + 37.68 + 2.83 = 45.22 \text{ radians}$$

$\therefore$ Total revolutions turned by the wheel $= \dfrac{45.22}{2\pi} = 7.2$ revolutions

EXAMPLE 15.11.

A flywheel had an initial angular speed of 3000 rev/min in clockwise direction. When a constant turning moment was applied to the wheel, it got subjected to a uniform anticlockwise angular acceleration of 3 rev/sec². Determine the angular velocity of the wheel after 20 seconds, and the total number of revolutions made during this period.

Solution: Initial angular velocity, $\omega_0 = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 3000}{60} = 314$ rad/s (clockwise)

Angular acceleration $\alpha = 3 \text{ rev/s}^2 = 3 \times 2\pi = 18.84 \text{ rad/s}^2$ (anticlockwise)

From the kinematic relation, $\omega = \omega_0 + \alpha t$, the angular velocity of the wheel after 20 seconds is

$$\omega = 314 + (-18.84) \times 20 = -62.8 \text{ rad/s}^2$$

The negative sign indicates that the wheel is turning anticlockwise.

Let the wheel change its direction of rotation after t_1 seconds. At the instant of change of direction, the velocity becomes zero. Accordingly

$$0 = 314 - 18.84 t_1 \quad ; \quad t_1 = \frac{314}{18.84} = 16.67 \text{ s}$$

For $t_1 = 16.67$ seconds, the wheel would turn clockwise and for

$t_2 = (20 - t_1) = (20 - 16.67)$

$= 3.33$ seconds, the wheel would turn anticlockwise.

Clockwise rotation $\theta_1 = \omega_0 t_1 + \frac{1}{2}\alpha t_1^2 = 314 \times 16.67 + \frac{1}{2}(-18.84) \times 16.67^2$

$= 2617.67$ radians

$= \dfrac{2617.67}{2\pi} = 416.83$ revolutions

Anticlockwise rotation $\theta_2 = 0 - \frac{1}{2}(-18.84) \times 3.33^2$

$= 104.46 \text{ radians} = \dfrac{104.46}{2\pi} = 16.63$ revolutions

$\therefore$ Gross number of revolutions $= 416.83 + 16.63 = 433.46$ rev

Net revolutions $= 416.83 - 16.63 = 400.20$ rev (clockwise)

EXAMPLE 15.12.

A flywheel rotating freely at 1800 rev/min in the clockwise direction is subjected to a variable counter clockwise torque which is intially applied at t = 0. The torque produces a counter-clockwise acceleration prescribed by the relation $\alpha = 3t$ rad/s² where t is the time in second. Make calculations for:

(a) *the time required for the flywheel to reduce its clockwise angular speed to 100 rev/min;*

(b) *the time required for the flywheel to reverse its direction of motion.*

Solution: Let it be presumed that the counter clockwise direction is positive.

$$\omega_1 = -\frac{2\pi \times 1800}{60} = -188.4 \text{ rad/s} \qquad \left(\because \omega = \frac{2\pi N}{60}\right)$$

$$\omega_2 = -\frac{2\pi \times 1000}{60} = -104.67 \text{ rad/s}$$

Angular acceleration $\alpha = \frac{d\omega}{dt} = 3t$ or $d\omega = 3t\,dt$. Integrating both sides within the prescribed limits.

$$\int_{-188.4}^{-104.67} d\omega = 3\int_0^t t\,dt$$

or

$$-104.67 + 188.4 = \frac{3t^2}{2} \quad \therefore \quad t = \sqrt{\frac{2}{3} \times 83.73} = 7.47 \text{ sec}$$

(*b*) At reversal of direction, the angular speed will be zero. Therefore,

$$\int_{-188.4}^{0} d\omega = 3\int t\,dt$$

or

$$188.4 = \frac{3t^2}{2} \quad \therefore \quad t = \sqrt{\frac{2}{3} \times 188.4} = 11.21 \text{ sec}$$

EXAMPLE 15.13.

A shaft is accelerated uniformly from 600 rev/min to 900 rev/min in 2 seconds. It continues accelerating at this rate for a further period of 4 seconds and then continues to rotate at the maximum speed attained. Make calculations for the time taken to complete the first 180 revolutions.

Solution: (*i*) During uniform acceleration from 600 rpm to 900 rpm (t = 2 sec)]

$$\omega_0 = \frac{2\pi \times 600}{60} = 62.8 \text{ rad/s} \qquad \left(\because \omega = \frac{2\pi N}{60}\right)$$

$$\omega_1 = \frac{2\pi \times 900}{60} = 94.2 \text{ rad/s}$$

$$\alpha_1 = \frac{\omega_1 - \omega_0}{t} = \frac{94.2 - 62.8}{2} = 15.7 \text{ rad/s}^2$$

The angular displacement θ_1 for the entire accelerating period of 6 seconds can be worked out by using the kinematic relation $\theta = \omega t + \frac{1}{2}\alpha t^2$. That is

$$\theta_1 = 62.8 \times 6 + \frac{1}{2} \times 15.7 \times 6^2 = 659.4 \text{ radians}$$

Since 1 revolution ≡ 2π radians,

Revolutions turned by the wheel during the accelerating period

$$= \frac{659.4}{2\pi} = 105 \text{ revolutions.}$$

(*ii*) Revolutions to be made by the shaft at uniform speed attained at the end of accelerating period = 180 − 105 = 75 revolutions

Corresponding angular displacement = 75 × 2π = 471 radians

Angular velocity at the end of acceleration period,

$$= \omega_0 + \alpha t = 62.8 + 15.7 \times 6 = 157 \text{ rad/s}$$

Time taken to turn 471 radians at uniform speed of 157 rad/s

$$t = \frac{\theta}{\omega} = \frac{471}{157} = 3 \text{ seconds}$$

$\therefore$ Required time = 2 + 4 + 3 = 9 seconds

EXAMPLE 15.14.

The motion of a gear is defined by the relation $\theta = 2t^3 - 5t^2 + 8t + 20$ where θ is in radians and t is in seconds.

Determine the angular displacement, the angular velocity and the angular acceleration at time t = 3 seconds.

Solution: Given that $\theta = 2t^3 - 5t^2 + 8t + 20$ ...(i)

Differentiating the above expression with respect to t, we get

$$\frac{d\theta}{dt} = 6t^2 - 10t + 8 \quad \text{or} \quad \omega = 6t^2 - 10t + 8 \qquad ...(ii)$$

where $\omega = \frac{d\theta}{dt}$ is the angular velocity of the rotating gear.

Differentiation of identity (*ii*) with respect to t gives:

$$\frac{d\omega}{dt} = 12t - 10 \quad \text{or} \quad \alpha = 12t - 10 \qquad ...(iii)$$

where α = angular acceleration of the rotating gear.

Then at t = 5 seconds we have:

Angular displacement $\theta = 2 \times (5)^3 - 5 \times (5)^2 + 8 \times 5 + 20$

$= 250 - 125 + 40 + 20 =$ 185 radians

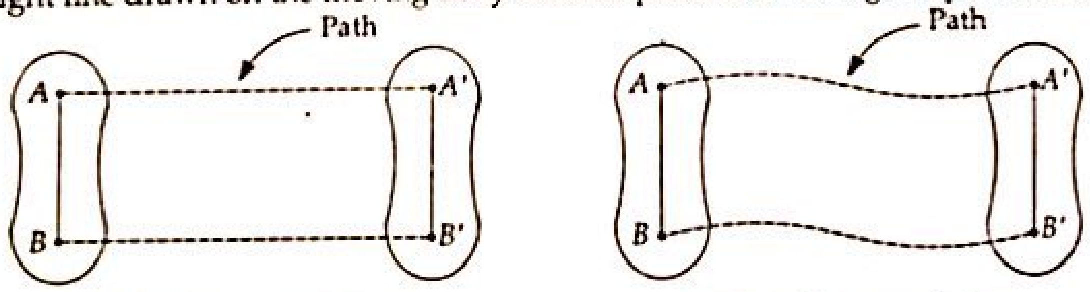
Angular velocity $\omega = 6 + (5)^2 - 10 \times 5 + 8 = 150 - 50 + 8 =$ 108 rad/s

Angular acceleration $\alpha = 12 \times 5 - 10 =$ 50 rad/s^2

15.3 GENERAL PLANE MOTION

Generally a body undergoes the following three types of plane motions :

- **Translation :** The particles constituting the rigid body move in parallel planes and travel the same distance. During translation, the particles have the same velocity and acceleration, and a straight line drawn on the moving body remains parallel to its original position at any time.



(a) Rectilinear translation *(b) Curvilinear translation*

Fig. 15.5

If the path traced by the particle during motion is a straight line, then the motion is said to be rectilinear translation. If the particle traces a curved path, the motion is called curvilinear translation.

- **Rotation :** The body rotates about a fixed point and all the particles constituting the body move in a circular path. The fixed point about which the body rotates is called the point of rotation and the axis passing through the point of rotation is called the axis of rotation. A point lying on the axis of rotation has a zero velocity and zero acceleration.

The point of rotation may lie inside the body (Fig. 15.6 *a*) or outside the body (Fig. 15.6 *b*)

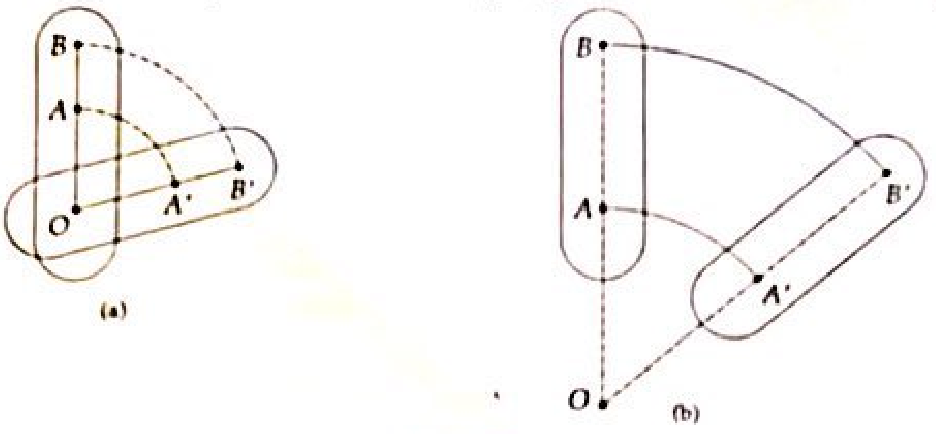


Fig. 15.6

• General plane motion (combined motion of translation and rotation) : There exist certain situations where a body possesses both motions of translation and rotation simultaneously at a particular instant. Examples of such a combined motion are :

— motion of roller without slipping

— motion of the wheel of a locomotive train, truck and car etc.

— a rod sliding against a wall at one end and floor at the other end.

The combination of rotation and translation motion is usually referred to as general plane motion.

With reference to Fig. 15.7, a link *AB* shifts to a new position at any instant. A link is a rigid piece of material joining two parts of a machine. This shift of the link can be considered in two parts:

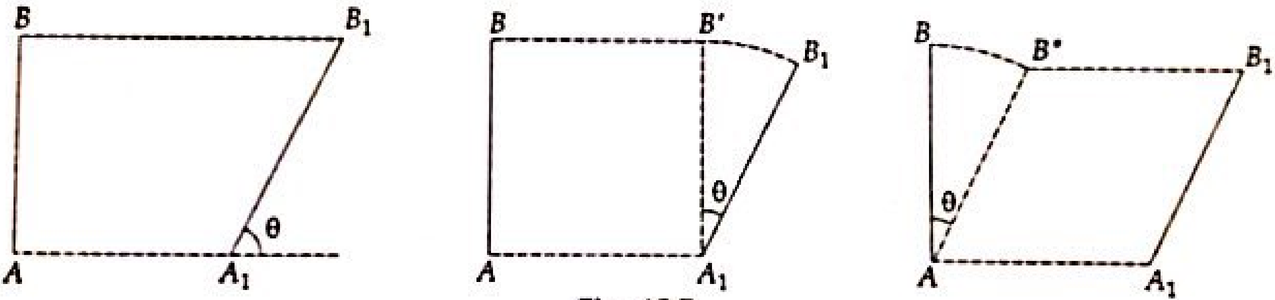


Fig. 15.7.

(*i*) The link translates by a distance AA_1 and takes the position A_1B'

(*ii*) From position A_1B', the link rotates by an angle θ and occupies the final position A_1B_1

OR

(*a*) The link rotates by an angle θ and takes the position AB''

(*b*) From position AB'', the link translates by distance AA_1 and shifts to the final position A_1B_1.

It is to be noted that the shift of the link from the initial to final position will remain same irrespective of the fact whether the motion of translation or that of rotation takes place first. Chasle's theorem states *that any general displacement of a rigid body can be represented by a combination of translatory motion and rotational motion.*

EXAMPLE 15.15.

Consider a wheel rolling on a horizontal surface, and a bar sliding against wall at one end and the floor at the other end. Illustrate how these acts can be split into motions of translation and rotation.

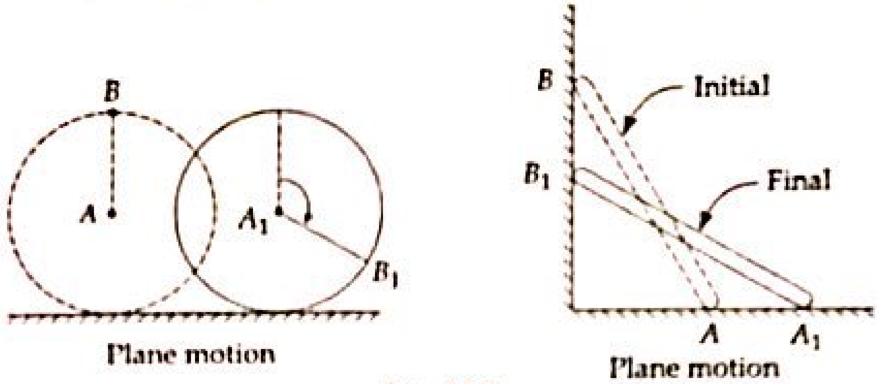


Fig. 15.8

Solution : Case (a) : Wheel rolling on a horizontal surface

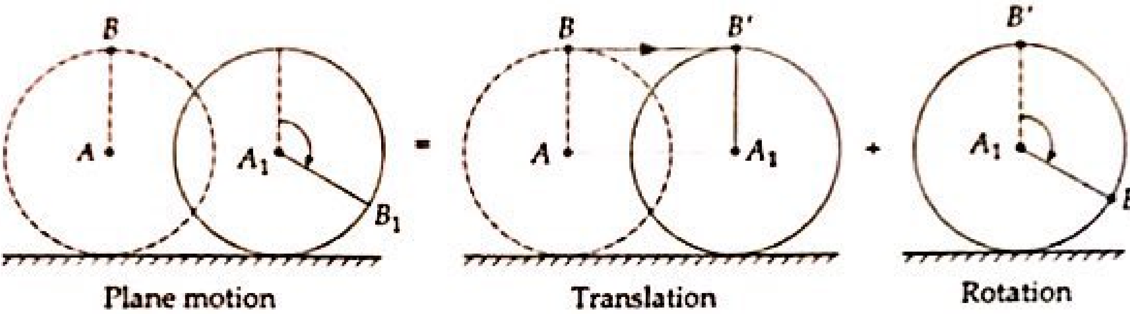


Fig. 15.9

With reference to Fig. 15.9, the general plane motion of a wheel rolling on a horizontal surface from initial position AB to final position A_1B_1 can be replaced by the following motions taken in order

(i) motion of translation from AB to A_1B'

(ii) rotation about A_1 from position A_1B' to A_1B_1

Case (b) : Bar sliding against wall at one end and floor at the other end

With reference to Fig. 15.10, the general plane motion of a bar sliding against wall at one end and floor at the other end from initial position AB to A_1B_1 can be replaced by the following motions taken in order

(i) motion of translation of the bar from position AB to A_1B'

(ii) rotation about A_1 from position A_1B' to A_1B_1

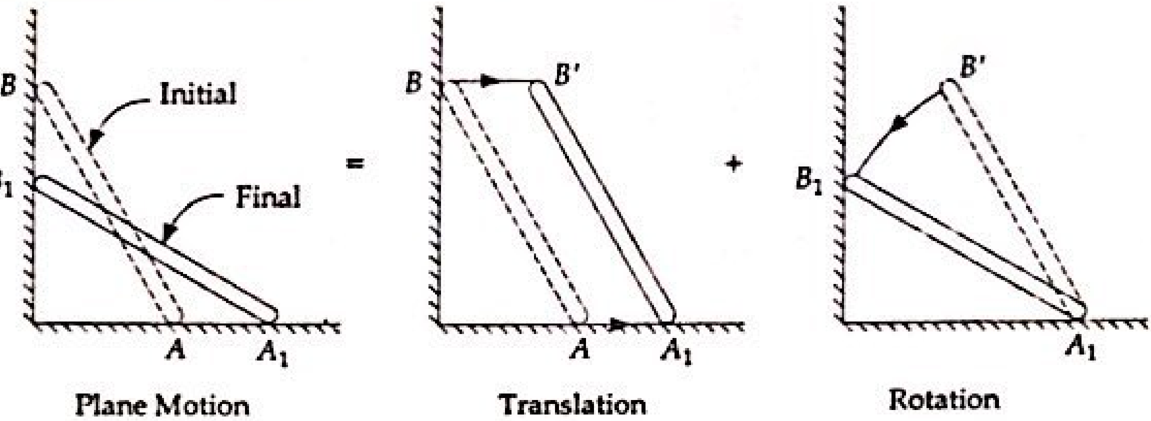


Fig. 15.10

15.4 INSTANTANEOUS CENTRE

While analysing plane motion of a body, a point can be located in the plane which has zero velocity. The plane motion of all the particles constituting the body may then be considered as pure rotation about that point. Such a point is called the *instantaneous centre* or *virtual centre of body*. The axis passing through this point and at right angles to the plane of motion is called *instantaneous axis of rotation*. The instantaneous centre changes every moment, and its locus is called centrode. The surface generated by the instantaneous axis is called the *axode*.

The following are the salient aspects in relation to instantaneous centre IC.

(*i*) The instantaneous centre is the point about which the body appears to rotate.

(*ii*) The instantaneous centre is not a fixed point but changes from one instant to another as the body rotates.

(*iii*) The velocity at an instantaneous centre is zero.

(*iv*) The instantaneous centre of a body may be inside or outside the body.

(*v*) The direction of velocity at a point on the body is normal to the line joining the point and the instantaneous centre.

(*vi*) The instantaneous centre of body lies at infinite distance when the velocities at any two points of the body are equal, parallel and in the same direction.

(*vii*) For a centre moving over a plane, instantaneous centre is the point of contact of the cylinder with the plane.

Depending on the given situation, following are some of the methods used to determine the instantaneous centre:

(*i*) Let v_a and ω be the linear and angular velocities of a point A on a rigid body (Fig. 15.11). Since ω and v_a are connected by the expression $v = \omega r$, the instantaneous centre I then lies at a distance $\frac{v_a}{\omega}$ along the perpendicular to the direction of velocity v_a at point A.

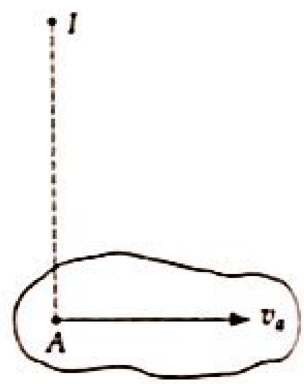


Fig. 15.11

$$IA = \frac{v_a}{\omega}$$

(*ii*) Let v_a and v_b be the linear velocities at points A and B on a rigid body (Fig. 15.12). These velocities are directed along the direction on AA' and BB' respectively. The instantaneous centre I is then the point of intersection of the lines erected perpendicular to the direction of velocities at the given points. Then

$$v_a = \omega \times IA \quad \text{and} \quad v_b = \omega \times IB$$

where ω is the angular velocity with which the body shall appear to rotate about the instantaneous centre I.

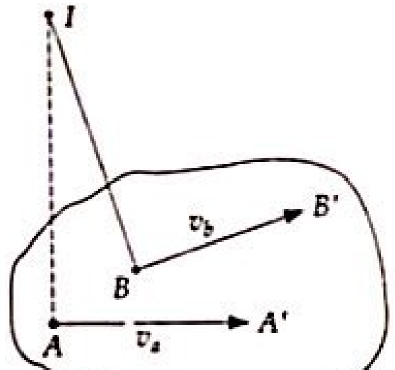


Fig. 15.12

(*iii*) Consider a solid circular cylinder in contact with two horizontal conveyer belts running with different velocities in the same or opposite directions as shown in Fig. 15.13.

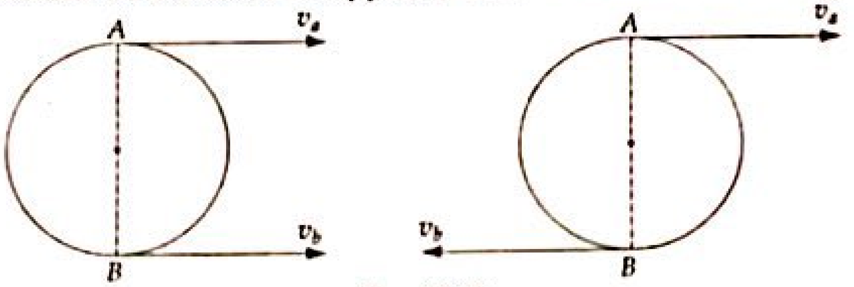


Fig. 15.13

The instantaneous centre is found by determining the point of intersection of line AB with the line joining the extremities of the velocities vectors (Fig. 15.14). It is to be noted that when the velocities v_a and v_b are in the same direction, I lies outside AB and divides it externally in the ratio v_a to v_b.

When velocities v_a and v_b are in opposite directions, I lies between A and B, and divides AB internally in the ratio v_a to v_b.

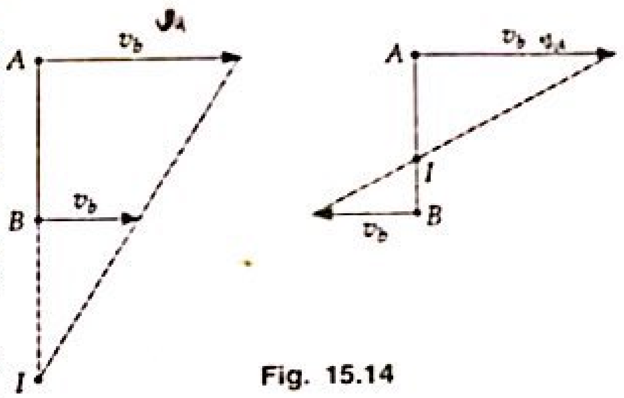


Fig. 15.14

Every point on the cylinder shall appear to rotate about the instantaneous centre I with an angular velocity ω, and therefore

$$\omega = \frac{v_a}{IA} = \frac{v_b}{IB}$$

(*iv*) For the piston-connecting rod assembly (Fig. 15.15)

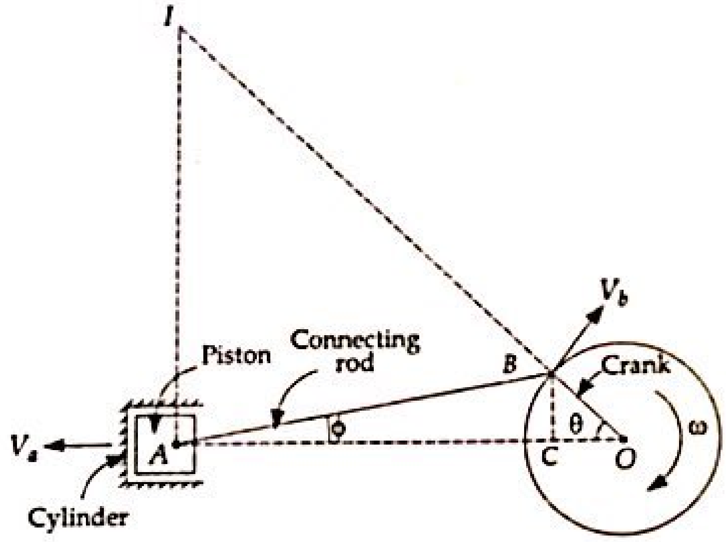


Fig. 15.15

The point B moves in a circle whose centre is O, and the direction of V_b is perpendicular to OB. The velocity V_a of the piston, at any instant, must be horizontal. The instantaneous centre I of the connecting rod AB is the point of intersection of perpendiculars to V_a and V_b through A and B respectively. Then angular velocity ω_{ab} of the connecting rod about I is

$$\omega_{ab} = \frac{V_a}{IA} = \frac{V_b}{IB} = \frac{\omega r}{IB} \quad ...(i)$$

where ω is angular velocity of the crank and r is its length. The distances IA and IB can be located as under :

$$AO = AC + CO = AB\cos\phi + BO\cos\theta = l\cos\phi + r\cos\theta$$

where l is the length of connecting rod.

$$IA = AO\tan\theta = (l\cos\phi + r\cos\theta)\tan\theta$$
$$= (l\cos\phi\tan\theta + r\sin\theta) \quad ...(ii)$$

$$IB = IO - BO = \frac{AO}{\cos\theta} - BO = \frac{l\cos\phi + r\cos\theta}{\cos\theta} - r$$

$$= \frac{l\cos\phi}{\cos\theta} + r - r = \frac{l\cos\phi}{\cos\theta} \quad ...(iii)$$

From the identities (*i*), (*ii*), (*iii*), the velocity V_a of the piston can be written as

$$V_a = \frac{IA}{IB}V_b = \frac{l\cos\phi\tan\theta + r\sin\theta}{l\cos\phi/\cos\theta} \times \omega r = \left(\cos\theta\tan\theta + \frac{r}{l}\frac{\sin\theta\cos\theta}{\cos\phi}\right)\omega r$$

Applying sine rule to ΔOAB,

$$\frac{OB}{\sin\phi} = \frac{AB}{\sin\theta}; \quad \frac{r}{\sin\phi} = \frac{l}{\sin\theta}; \quad \frac{r}{l} = \frac{\sin\phi}{\sin\theta}$$

Then $\quad V_a = \omega(r\sin\theta + r\cos\theta\tan\phi) = \omega(l\sin\phi + r\cos\theta\tan\phi) \quad ...(15.4)$

EXAMPLE 15.16

At the instant shown in Fig. 15.16, the rod AB is rotating clockwise at 2.5 rad/sec. If end C of the rod BC is free to move on a horizontal surface, make calculations for the angular velocity of rod BC and the velocity of its end point C.

Solution : Velocity of end B, $V_b = \omega r = 2.5 \times 1.5 = 3.75$ m/s

The direction of V_b is normal to AB and as BC is perpendicular to AB, it lies along BC. The velocity V_c of end C, at any instant, must be horizontal.

The instantaneous centre I of rod BC is the point of intersection of perpendiculars to V_b and V_c through B and C respectively (Fig. 15.17)

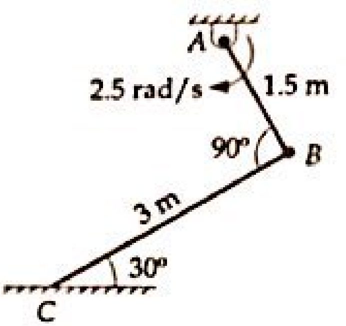


Fig. 15.16

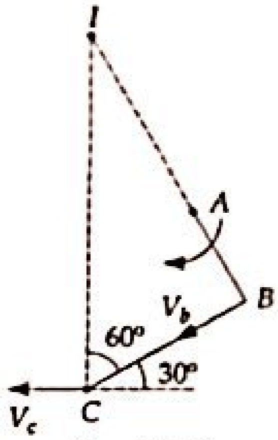


Fig. 15.17

$$IC = \frac{BC}{\cos 60^\circ} = \frac{3}{0.5} = 6 \text{ m}$$

$$IB = IC \sin 60° = 6 \times 0.866 = 5.196 \text{ m}$$

Angular velocity of rod BC,

$$\omega = \frac{V_b}{IB} = \frac{V_c}{IC}$$

$$\omega = \frac{3.75}{5.196} = 0.722 \text{ rad/s (clockwise)}$$

and $$V_c = \omega \times IC = 0.722 \times 6 = 4.33 \text{ m/s}$$

EXAMPLE 15.17.

In a crank and connecting rod mechanism, the crank is 300 mm long and the connecting rod 1500 mm long. If the crank rotates uniformly at 300 rpm, find the velocity of the cross-head when the crank is inclined at 30° with the inner dead centre.

Solution: Refer Fig. 15.18 for the crank and connecting rod mechanism.

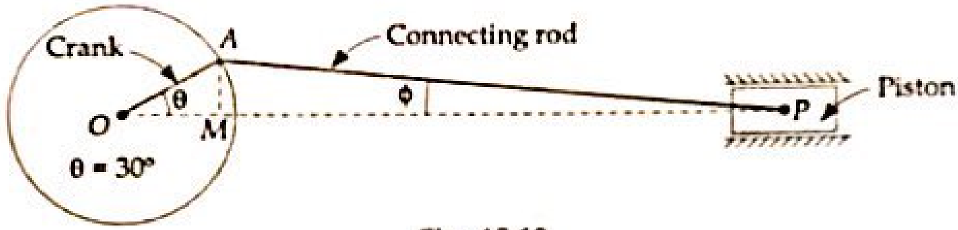


Fig. 15.18

Crank radius r = 300 mm

Length of connecting rod l = 1500 mm

From the law of triangles: $\frac{l}{\sin\theta} = \frac{r}{\sin\phi}$

$$\sin\phi = \frac{r\sin\theta}{l} = \frac{300 \times \sin 30°}{1500} = 0.1$$

That gives: $\phi = 5.74°$

Let x denote the distance between the crank centre O and the piston point P. Then

$$x = OM + OP = r\cos\theta + l\cos\phi$$

$$\frac{dx}{dt} = r\sin\theta \times \frac{d\theta}{dt} + l\sin\phi \times \frac{d\phi}{dt}$$

$$= r\sin\theta \frac{d\theta}{dt} + l\sin\phi \frac{d\phi}{d\theta} \times \frac{d\theta}{dt}$$

$$= r\sin\theta \times \omega + l\sin\phi \times \left(\frac{r}{l}\frac{\cos\theta}{\cos\phi}\right) \times \omega \quad \text{...(a)}$$

Angular velocity $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 31.4 \text{ rad/s}$

Substituting the appropriate values in expression (*a*), we get velocity of cross-head,

$$\frac{dx}{dt} = 300\sin 30° \times 31.4 + 1500 \times \sin 5.74° \times \left(\frac{300}{1500} \times \frac{\cos 30°}{\cos 5.74°}\right) \times 31.4$$

$$= 4710 + 820 = 5530 \text{ mm/s} = 5.53 \text{ m/s}$$

EXAMPLE 15.18.

In a reciprocating engine mechanism (Fig. 15.21), the lengths of the crank OB and connecting rod AB are 30 cm and 1 metre respectively. If the crank is rotating at a constant angular velocity of 200 rev/min, determine (a) the angular velocity of connecting rod and (b) the velocity of the piston when the crank makes an angle of $\theta = 45°$ with the horizontal.

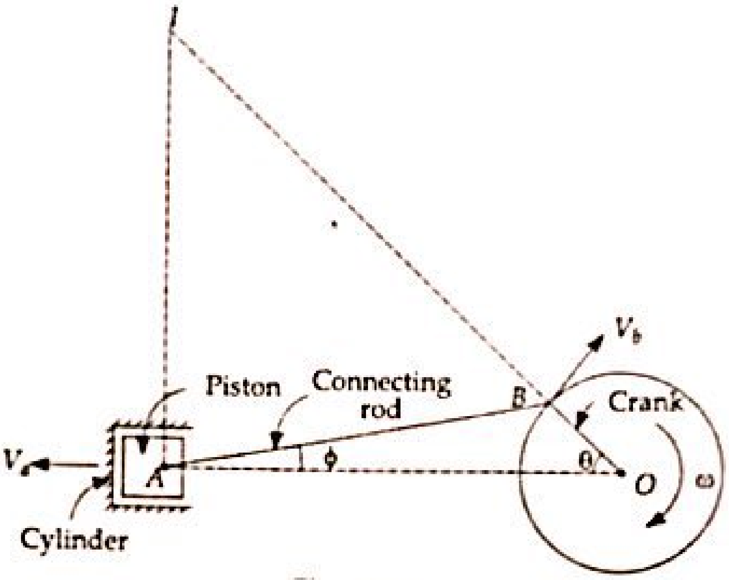


Fig. 15.19

Solution : Angular velocity of crank $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 200}{60} = 20.93$ rad/s

$$V_b = \omega r = 20.93 \times 0.3 = 6.279 \text{ m/s}$$

The point B moves in a circle whose centre is O, and the direction of V_a is perpendicular to OB. The velocity V_a of the piston, at any instant must be horizontal. The instantaneous centre I of the connecting rod AB is the point of intersection of perpendiculars to V_a and V_b through A and B respectively. Then angular velocity ω_{ab} of the connecting rod about I is

$$\omega_{ab} = \frac{V_a}{IA} = \frac{V_b}{IB} = \frac{\omega r}{IB} \quad \text{...}(i)$$

Let ϕ be the angle between the connecting AB and the line AO. Then applying sine rule to ΔOAB

$$\frac{OB}{\sin\phi} = \frac{AB}{\sin\theta}; \quad \sin\phi = \frac{OB}{AB}\sin\theta = \frac{0.3}{1} \times \sin 45° = 0.2121$$

$$\therefore \quad \phi = 12.25°$$

The distance IA and IB are worked out from the relations

$$IA = l\cos\phi \tan\theta + r\sin\theta = (1 \times \cos 12.25° \times \tan 45°) + 0.3\sin 45°$$

$$= 0.9772 + 0.2121 = 1.1843 \text{ m}$$

$$\text{and} \quad IB = \frac{l\cos\phi}{\cos\theta} = 1 \times \frac{\cos 12.25°}{\cos 45°} = 1.382 \text{ m}$$

Then from identity (*i*)

angular velocity of connecting rod, $\omega_{ab} = \dfrac{6.279}{1.382} = 4.543$ rad/s

velocity of piston $V_a = \omega_{ab} \times IA = 4.543 \times 1.1843 = 5.38$ m/s

EXAMPLE 15.19.

A cylinder of diameter 3 m rolls without slipping along a horizontal surface PQ as shown in Fig. 15.20. If its centre has a uniform velocity of 30 m/s, determine the velocities of the points B and D lying on the rim of the cylinder.

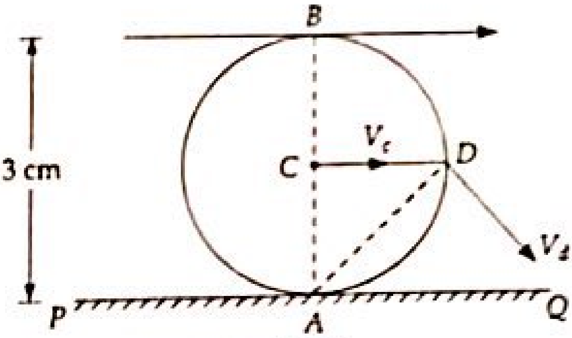


Fig. 15.20

Solution : When the cylinder rolls without slipping, its point of contact with the horizontal surface at any instant has zero velocity. Thus the point A is instantaneously at rest and every other point on the cylinder has angular velocity about A. Hence A is instantaneous centre of the cylinder. Then

$$V_c = \omega \times AC; \quad V_B = \omega \times AB; \quad V_d = \omega \times AD$$

$$\text{Angular velocity of the cylinder } \omega = \frac{V_c}{AC} = \frac{30}{3/2} = 20 \text{ rad/s}$$

$$\text{Distance } AD = \sqrt{(DC)^2 + (AC)^2} = \sqrt{1.5^2 + 1.5^2} = 2.121 \text{ m}$$

$\therefore$ Velocity of point B, $V_b = \omega \times AB = 20 \times 3 = 60$ m/s

Velocity of point D, $V_d = \omega \times AD = 20 \times 2.121 = 42.42$ m/s

EXAMPLE 15.20.

A 30 cm diameter cylinder rolls without slipping along a horizontal floor. The cylinder has its contact with the floor at point D, and its centre point C has a uniform velocity of 4.5 m/s. Determine the velocities of points A and B on the circumference of the cylinder (Fig. 15.21).

Solution : Choosing centre of cylinder C as the pole, angular velocity ω of the wheel is

$$\omega = \frac{V_c}{r} = \frac{4.5}{0.15} = 30 \text{ rad/s}$$

Velocity $\vec{V}_A$ at point A

= velocity of C + velocity of A with respect to C

$$= \vec{V}_C + \overrightarrow{(\omega \times CA)} = \overrightarrow{4.5} + \overrightarrow{(30 \times 0.15)}$$

$$= \overrightarrow{4.5} + \overrightarrow{4.5} = \sqrt{4.5^2 + 4.5^2} = 6.36 \text{ m/s}$$

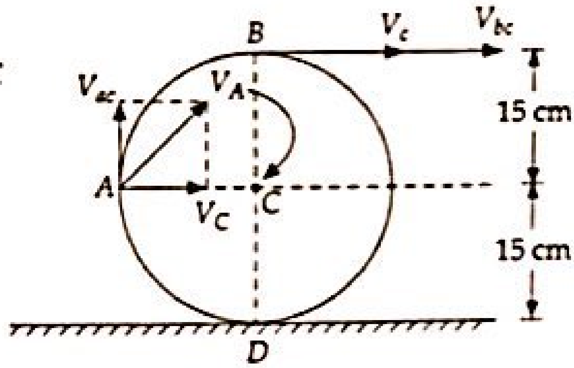


Fig. 15.21

The inclination α of this velocity with horizontal is

$$\alpha = \tan^{-1}\left(\frac{4.5}{4.5}\right) = 45°$$

Likewise: $\vec{V}_B = \vec{V}_c + \vec{V}_{bc} = \vec{V}_c + \overrightarrow{(\omega + CB)}$

$$= \overrightarrow{4.5} + \overrightarrow{(30 \times 0.15)} = \overrightarrow{4.5} + \overrightarrow{4.5} = 9.0 \text{ m/s}$$

Here addition has been done mathematically becuase both the velocity components lie along the same direction. The velocity vector V_B is along the horizontal direction.

Alternatively: The cylinder rolls wihtout slipping and accordingly at any instant, the contact point D has zero velocity and represents the instantaneous centre. Therefore,

$$V_A = \omega \times (DA) = 30 \times \sqrt{0.15^2 + 0.15^2} = 30 \times 0.212 = 6.36 \text{ m/s}$$

The velocity vector V_A is perpendicular to DA and at 45° to the horizontal.

and $V_B = \omega \times (DB) = 30 \times 0.3 = 9.0$ m/s.

The velocity vector V_B is perpendicular to DB, *i.e.*, along the horizontal.

EXAMPLE 15.21.

The ends of a slender beam AB of length 2.5 m are constrained to remain in contact with a horizontal floor and a vertical wall respectively as shown in Fig. 15.22. If the end A has a velocity of 1.5 m/s, determine the angular velocity of the beam and the velocity of its end B at the position shown in the figure.

Solution: The velocity V_a at end A is horizontal and the velocity V_b at end B is vertical. The instantaneous centre I of the beam AB is the point of intersection of perpendicular to V_a and V_b through A and B respectively.

$$IA = AB \sin 60° = 2.5 \times 0.866 = 2.165 \text{ m}$$

$$IB = AB \cos 60° = 2.5 \times 0.5 = 1.25 \text{ m}$$

Then angular velocity ω_{ab} of the beam is given by

$$\omega_{ab} = \frac{V_a}{IA} = \frac{1.5}{2.165} = 0.693 \text{ rad/s}$$

∴ Velocity of end B is

$$V_b = \omega_{ab} \times IB = 0.693 \times 1.25 = 0.866 \text{ m/s}$$

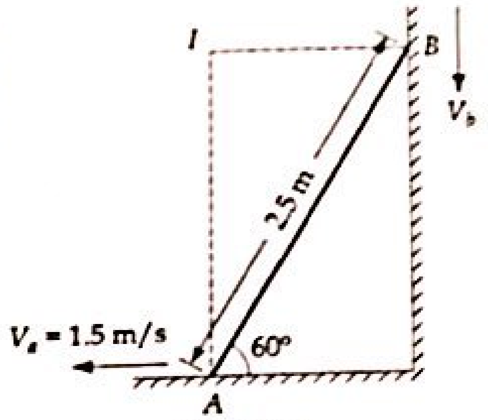


Fig. 15.22

EXAMPLE 15.22.

What is kinetics? Define the terms impulse and angular momentum.

A bar AB rests at the edge of a wall at some point C with its end A resting on a horizontal floor as shown in Fig. 15.23. If the end A moves with a constant velocity V_a, set up an expression for the angular velocity of the bar in terms of h and θ. Proceed to calculate the angular velocity if

$$V_a = 2.5 \text{ m/s}; \quad h = 4 \text{ m} \quad \text{and} \quad \theta = 30°$$

Solution: The velocity of the end A is along the horizontal line OA and the velocity of C on the bar at any instant is along the bar. The position of instantaneous centre I is determined by drawing lines perpendicular to the direction of these velocities.

The angular velocity of the bar $\omega = \dfrac{V_a}{IA}$ ...(*i*)

From triangles OAC and AIC, we have

$$AC = \frac{OC}{\sin\theta};$$

$$AC = IA\cos(90 - \theta) = IA\sin\theta$$

$$\therefore \quad \frac{OC}{\sin\theta} = IA\sin\theta$$

or $$IA = \frac{OC}{\sin^2\theta} = \frac{h}{\sin^2\theta}$$

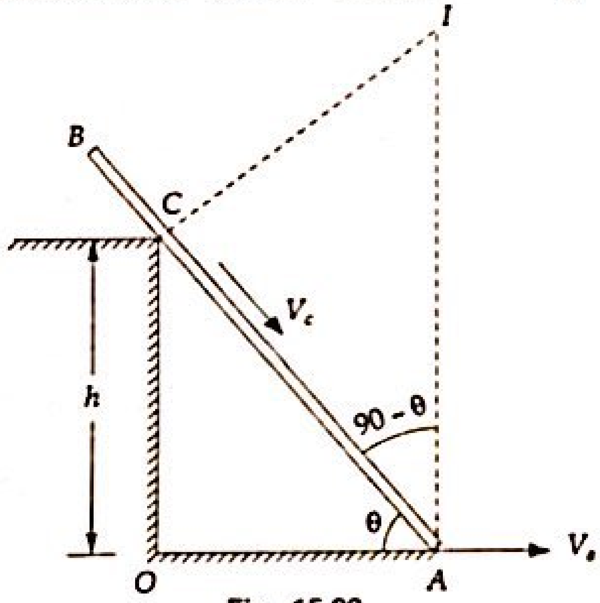


Fig. 15.23

Substituting $IA = \frac{h}{\sin^2\theta}$ in expression (*i*), we get

$$\omega = \frac{V_a \sin^2\theta}{h} = \frac{2.5\sin^2 30}{4} = 0.1562 \text{ rad/s}$$

EXAMPLE 15.23.

A bar of length 2 m has its ends A and B constrained to move horizontally and vertically as shown in Fig. 15.22. The end A moves with a constant velocity of 6 m/s horizontally. Make calculations for:

(*a*) *the angular velocity of the bar AB,*

(*b*) *the velocity of the end B, and*

(*c*) *the velocity of the midpoint of the bar at the instant when the bar makes an angle of 30° with the horizontal.*

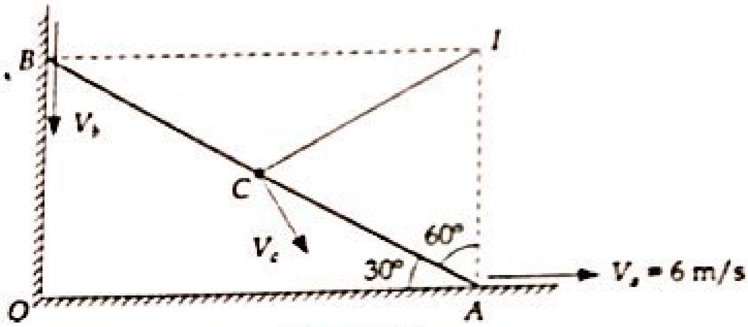
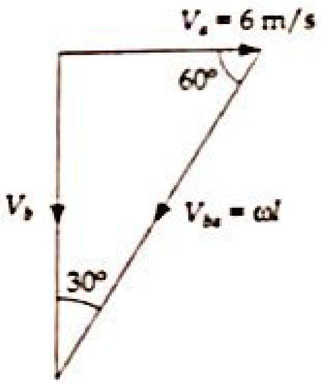


Fig. 15.24

Solution: Velocity of end A, $V_a = 6$ m/s

Velocity of end B, $V_b = V_a + V_{ba}$

where V_{ba} is the relative velocity of end B with respect to end A and it is due to rotation of bar AB about A. This relative is normal to bar AB at point B and it equals ωl.



From the vector diagram,

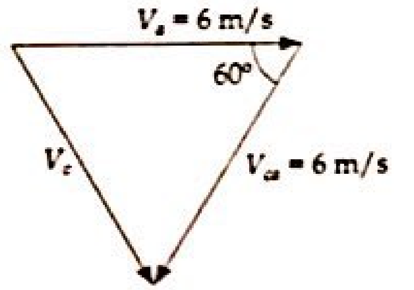
$$\frac{V_a}{\sin 30°} = \frac{V_{ba}}{\sin 90°} = \frac{V_b}{\sin 60°}$$

(*i*) $V_b = \frac{V_a}{\sin 30°} \times \sin 60° = \frac{6}{0.5} \times 0.866$

$= 10.39$ m/s

(*ii*) $V_{ba} = \frac{V_a}{\sin 30°} \times \sin 90° = \frac{6}{0.5} \times 1 = 12$ m/s

$\therefore$ Angular velocity of bar, $\omega = \frac{V_{ba}}{l} = \frac{12}{2} = 6$ rad/s



(*iii*) Velocity of midpoint C of the bar

$$\vec{V}_c = \vec{V}_a + \vec{V}_{ca}$$

$V_a = 6$ m/s and $V_{ca} = \omega \frac{l}{2} = 6 \times \frac{2}{2} = 6$ m/s

With reference to vector diagram, the magnitude of velocity V can be determined by using the cosine relation

$$V_c = \sqrt{V_a^2 + V_{ca}^2 - 2V_a V_{ca} \cos 60°}$$

$$= \sqrt{6^2 + 6^2 - 2 \times 6 \times 6 \times 0.5} = \sqrt{36} = 6 \text{ m/s}$$

This is also apparent from the vector diagram which is an equilateral triangle.

Alternatively the problem can be worked out by the instantaneous centre method.

The velocity V_a at end A is horizontal and the velocity V_b at end B is vertical. The instantaneous centre I of the bar AB is the point of intersection of perpendicular to V_a and V_b through A and B respectively

$$IA = BO = AB\sin 30° = 2\sin 30° = 1 \text{ m}$$
$$IB = AO = AB\cos 30° = 2\cos 30° = 1.732 \text{ m/s}$$

Then the angular velocity of the bar is given by

$$\omega_{ab} = \frac{V_a}{IA} = \frac{6}{1} = 6 \text{ rad/s}$$

(*i*) Velocity of end $B = \omega \times IB = 6 \times 1.732$
$= 10.392$ m/s

(*ii*) In triangle IAC; $AC = 1$ m; $IA = 1$ m (calculated above) and $\angle IAC = 60°$

Obviously the triangle IAC is an equilibrium triangle. That gives

$$IC = IA = 1 \text{ m}$$

∴ Velocity of mid point $V_c = \omega \times IC = 6 \times 1 = 6$ m/s

EXAMPLE 15.24.

The ends A and B of line 5 m long are restrained to move in vertical and horizontal guides as shown in Fig. 15.25. The end A moves 2.5 m/s upward when it is 3 m above C. Determine:

(*i*) *angular velocity of link AB,*
(*ii*) *velocity of end B, and*
(*iii*) *relative velocity of end B with respect to end A.*

Solution: Velocity of end A: $V_a = 2.5$ m/s
Velocity of end B: $V_b = V_a + V_{ba}$

where V_{ba} is the relative velocity of end B with respect to end A and it is due to rotation of link about A and it equals ωl.

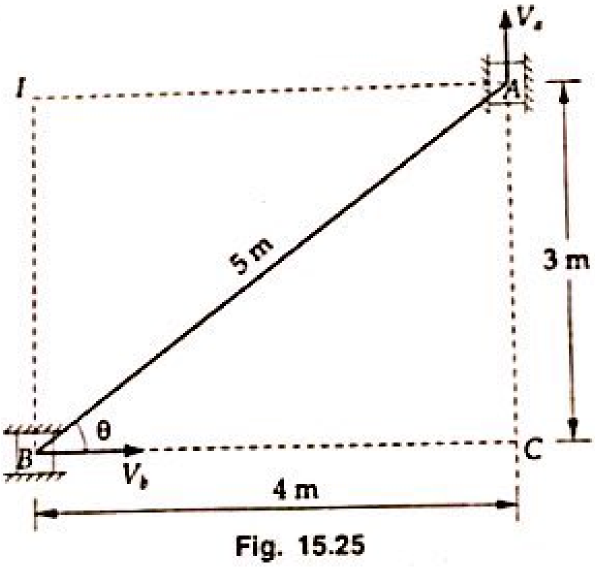
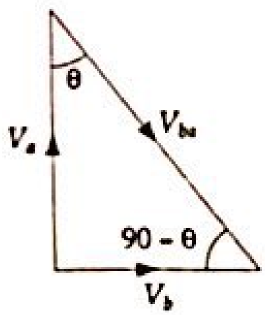


Fig. 15.25

From the vector diagram,



$$\frac{V_a}{\sin(90-\theta)} = \frac{V_{ba}}{\sin 90°} = \frac{V_b}{\sin\theta}$$

or

$$\frac{V_a}{\cos\theta} = \frac{V_{ba}}{\sin 90°} = \frac{V_b}{\sin\theta}$$

From the given geometry: $\sin\theta = \frac{3}{5} = 0.6$ and $\cos\theta = \frac{4}{5} = 0.8$

Then: (i) $V_b = \frac{V_a}{\cos\theta} \times \sin\theta = \frac{2.5}{0.8} \times 0.6 = 1.875$ m/s

(ii) $V_{ba} = \frac{V_a}{\cos\theta} \times \sin 90^\circ = \frac{2.5}{0.8} \times 1 = 3.125$ m/s

(iii) Angular velocity of bar,

$$\omega_{ab} = \frac{V_{ba}}{l} = \frac{3.125}{5} = 0.625 \text{ rad/s}$$

Alternatively, the problem can be worked out by instantaneous centre method

The velocity at end A is horizontal and that at end B is vertical. The point of intersection of the normals to these velocities V_a and V_b locates the instantaneous centre I.

$$IA = 4 \text{ m} \quad \text{and} \quad IB = 3 \text{ m}$$

Then the angular velocity of the bar is given by

$$\omega_{ab} = \frac{V_a}{IA} = \frac{2.5}{4} = 0.625 \text{ rad/s}$$

Velocity of end $B = \omega \times IB = 0.625 \times 3 = 1.875$ m/s

Velocity of end B relative to end A,

$$V_{ba} = \omega \times l = 0.625 \times 5 = 3.125 \text{ m/s}$$

EXAMPLE 15.25.

A compound wheel of configuration illustrated in Fig. 15.26 rolls without slipping on a horizontal surface PQ. If the velocity at the centre of wheel is 1 m/s, determine the velocities at the points A, B and D indicated in the figure.

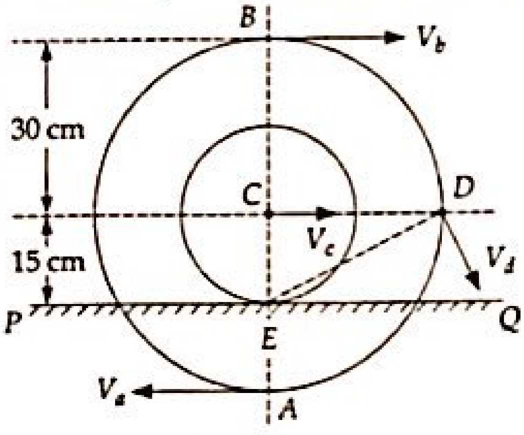


Fig. 15.26

Solution : When the wheel rolls without slipping, its point of contact with the horizontal surface at any instant has zero velocity. Thus the point E is instantaneously at rest and every other point on the wheel has angular velocity about E. Hence E is the instantaneous centre of the wheel. Then

$$V_c = \omega \times CE; \quad V_a = \omega \times AE$$
$$V_b = \omega \times BE \quad \text{and} \quad V_d = \omega \times DE$$

Angular velocity of the wheel $\omega = \frac{V_c}{CE}$

$$= \frac{1}{0.15} = 6.67 \text{ rad/s}$$

Velocity of point A: $V_a = \omega \times AE = 6.67 \times 0.15 = 1$ m/s

Velocity of point B: $V_b = \omega \times BE = 6.67 \times 0.45 = 3$ m/s

$$\text{Distance } DE = \sqrt{(CE)^2 + (CD)^2}$$

$$= \sqrt{(0.15)^2 + (0.3)^2} = 0.364 \text{ m}$$

$\therefore$ Velocity of point $D: V_d = \omega \times DE = 6.67 \times 0.364 = 2.43$ m/s

The directions of these velocities at the given points would be as indicated in the figure.

EXAMPLE 15.26.

A cylindrical roller, 50 cm in diameter, is in contact with two horizontal conveyor belts running at uniform speeds of 5 m/s and 3 m/s as shown in Fig 15.27.

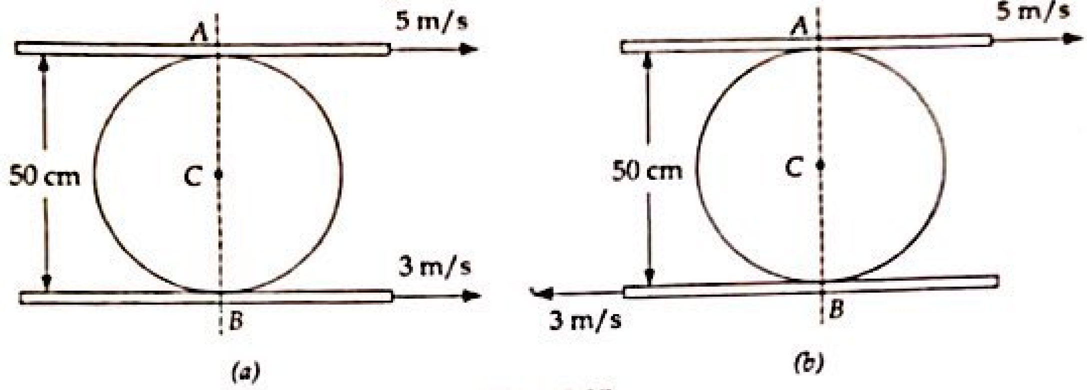


Fig. 15.27

Assuming that there is no slip at the points of contact, determine

(i) the position of the instantaneous centre of the roller,

(ii) the linear velocity of the centre C, and

(iii) the angular velocity of the roller.

(b) How these parameters would be affected if the velocities of the belts are in opposite direction (Fig. 15.27 b)

Solution : Case (*a*) : Velocities V_a and V_b are in the same direction.

The instantaneous centre is found by determining the point of intersection of line *AB* with the line joining the extremities of the velocity vectors (Fig. 15.28), then

$$V_a = \omega_{ab} \times IA; \quad V_b = \omega_{ab} \times IB$$

or $$\frac{IA}{IB} = \frac{V_a}{V_b} = \frac{5}{3}$$

or $$\frac{IB + AB}{IB} = \frac{5}{3}$$

or $$\frac{IB + 0.5}{IB} = \frac{5}{3}$$

∴ $$5IB - 3IB = 3 \times 0.5$$

or $$IB = \frac{3 \times 0.5}{2} = 0.75 \text{ m} = 75 \text{ cm}$$

$$IA = IB + AB = 75 + 50 = 125 \text{ cm}$$

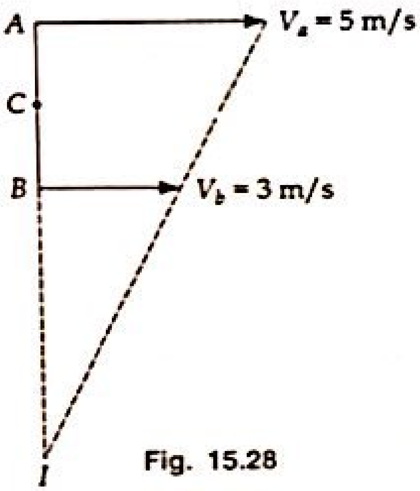


Fig. 15.28

Angular velocity of the roller = angular velocity of *A* about *I*

$$\omega = \frac{V_a}{IA} = \frac{5}{1.25} = 4 \text{ rad/s}$$

Linear velocity of the centre of roller,

$$V_c = \omega \times IC = 4 \times (1.25 - 0.25) = 4 \text{ m/s}$$

(*b*) When velocities V_a and V_b are in opposite directions, *I* lies between *A* and *B* (Fig. 15.29), and divides *AB* internally in the ratio V_a to V_b. That is,

$$\frac{IA}{IB} = \frac{V_a}{V_b} = \frac{5}{3}$$

or $\frac{0.5 - IB}{IB} = \frac{5}{3}$;

$$IB = 0.1875 \text{ m} = 18.75 \text{ cm}$$

$$IA = 50 - 18.75 = 31.25 \text{ cm}$$

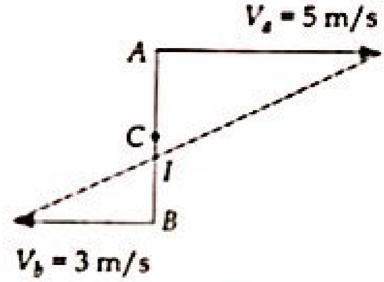


Fig. 15.29

Angular velocity of the roller

$$\omega = \frac{V_a}{IA} = \frac{5}{0.3125} = 16 \text{ rad/s}$$

Linear velocity of the centre of roller,

$$V_c = \omega \times IC = \omega \times (IA - AC) = 16 \times (0.3125 - 0.25) = 1 \text{ m/s}$$

15.5 CORIOLIS' LAW

If a point moves along a path that has a rotational motion, the absolute acceleration of the point is given by the vector sum of three accelerations:

(*i*) acceleration of the coincident point relative to which the point under consideration is moving,

(*ii*) acceleration of the point relative to the coincident point, and

(*iii*) coriolis component of acceleration

We know that any plane motion can be replaced by a translation defined by the motion of an arbitrary reference point O and simultaneous rotation about O. Then the absolute acceleration of any point A can be written as

$$\vec{a}_A = \vec{a}_O + \vec{a}_{AO}$$

That is, the absolute acceleration of point A is the sum of its acceleration $\bar{a}_{AO}$ relative to the frame of translation and the acceleration $\bar{a}_O$ of a point of that frame. Now, if the frame is made to rotate with an angular velocity ω, then

$$\vec{a}_A = \vec{a}_B + \vec{a}_{A\omega} + \vec{a}_C$$

where $\vec{a}_A$ = absolute acceleration of particle at point A

$\vec{a}_B$ = acceleration of point B of moving frame ω coinciding with A

$\vec{a}_{A\omega}$ = acceleration of A relative to moving frame ω, and

$\vec{a}_C$ = Coriolis acceleration also called the complementary acceleration

Consider a collar (link 2) which slides at constant velocity along a rod OM (link 1) towards point Q. Simultaneously the rod rotates at a constant angular velocity ω about O as shown in Fig. 15.30. At an instant, the point A on the rod is coincident with point B on the collar. When the collar moves on the rod to point Q in the dt, the rod rotates through an angle $d\theta$ during the same time interval. Then the final position of point A is A_1 and that of point B is B_2.

The motion of the collar from B to B_2 can be considered to be accomplished in the following three stages:

(*i*) B to A_1 due to rotation of rod

(*ii*) A_1 to B_1 due to outward velocity of collar on the rod, and

(*iii*) B_1 to B_2 due to coriolis component of acceleration

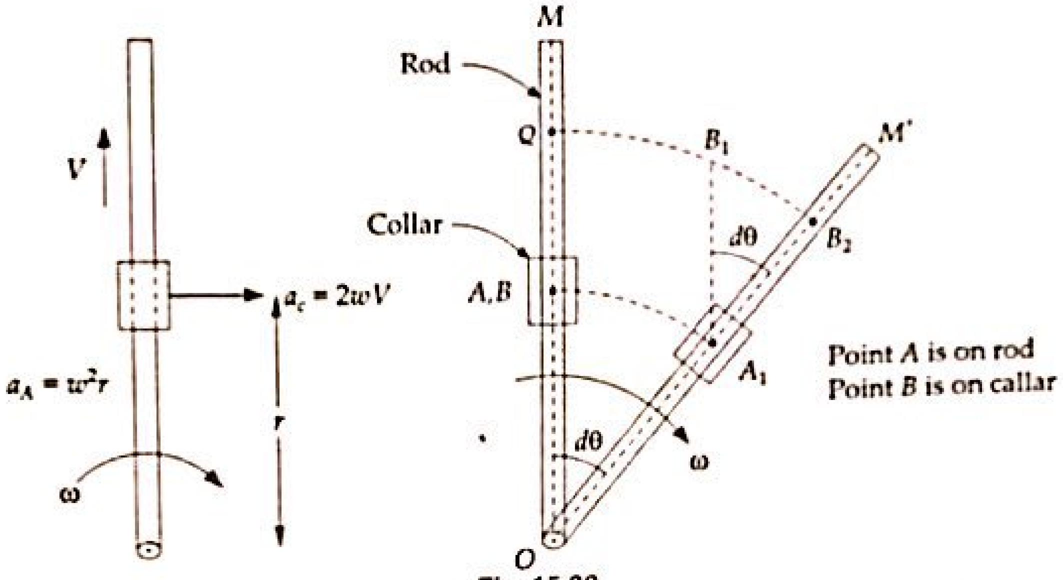


Fig 15.30

The coriolis component acts in the direction of relative velocity vector for the two coincident points (A and B) rotated by 90° in the direction of angular velocity of the rotation of rod, and it is given by

$$\vec{a}_C = 2\omega V_{ba}$$

where V_{ba} is the velocity of B relative to A

$$\text{Coriolis force} = m \times \vec{a}_C$$

where m is the mass of the collar

It is worthwhile to mention that:

(i) The angular velocity of the rod is constant and hence $\vec{a}_A$ reduces to its normal component of magnitude $\omega^2 r$

(ii) Since the collar slides along the rod with constant relative speed, the relative acceleration is zero.

(iii) Coriolis acceleration is a vector perpendicular to rod OM and is of magnitude $2\omega V_{ba}$ and is directed as shown.

Obviously the acceleration of the collar consists of two vectors $\vec{a}_C$ and $\vec{a}_A$

The concept of coriolis acceleration is very useful in the study of long range projectiles and of the bodies whose motions are appreciably affected by the rotation of earth.

REVIEW AND SUMMARY

1. In a circular (rotary) motion, the movement of the particle is along a circular path. The particle repeats its journey along the same circular path about the centre of rotation which remains fixed.

   The shafts, pulleys and flywheels etc. undergo circular motion when they rotate about their geometric axis.

2. The rate of change of angular displacement of a body is called angular velocity.

$$\omega = \frac{d\theta}{dt}$$

If the body turns N revolutions per minute, then

$$\omega = \frac{2\pi N}{60} \text{ rad/s}$$

The linear velocity V and the angular velocity ω are related by the expression: $V = \omega r$

3. The rate of change of angular velocity of a body is called angular acceleration.

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

The linear acceleration a and the angular acceleration α are related by the expression: $a = \alpha r$.

4. When a body moves along a curved path, the linear velocity is referred to as tangential velocity. Further, the body undergoes tangential and normal components of acceleration given by

$$a_t = \alpha r \quad \text{and} \quad a_n = \frac{v^2}{r} = \omega^2 r$$

The tangential acceleration is due to change in magnitude of velocity and the normal acceleration is due to change in direction of the body.

5. When a body moves in a circular path with uniform acceleration, the equations of motion are :

$$\omega = \omega_0 + \alpha t \,;\ \omega^2 - \omega_0^2 = 2\,\alpha\theta$$

and $$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

where ω_0 is the initial angular velocity, ω is the final angular velocity, α is angular acceleration of the body and θ represents the angle moved in time t.

6. A body is said to have motion of translation when the particles constituting the body move in parallel planes and travel the same distance.

7. A body is said to have motion of rotation when the body moves about a fixed point and all the particles constituting the body move in a circular path. The fixed point about which the body rotates is called the point of rotation and the axis passing through the point of rotation is called the axis of rotation. A point lying on the axis of rotation has a zero velocity and zero acceleration.

8. The combination of rotation and translation motion is usually referred to as general plane motion. The body can then be assumed to be rotating about a certain point which is known as instantaneous centre of rotation.

9. *Chasle's theorem* states that any general displacement of a rigid body can be represented by a combination of translatory motion and rotational motion.

REVIEW QUESTIONS

A. Conceptual and conventional questions :

1. State the difference between curvilinear motion and circular (rotary motion)
2. Define the following terms associated with circular motion :
   angular displacement, angular velocity and angular acceleration
3. Set up the $a = \alpha r$ relation between linear acceleration (a) and angular acceleration (α) where r is the distance of the body from the centre of rotation.
4. Explain the concept of general plane motion. List some examples of such a motion.

5. State and explain Chasle's theorem.

6. Define the term instantaneous centre or virtual centre in plane motion of rigid bodies.

7. Set up a relation for the velocity of position of a reciprocating heat engine in terms of angular speed of crank, crank angle and ratio of lengths of connecting rod and that of crank

8. A flywheel starting from rest rotates at uniform angular acceleration, and after having rotated 400 revolutions its angular velocity is 600 rpm. Find the angular acceleration and the interval of time to complete 400 revolutions. (0.785 rad/s^2, 80 sec)

9. The speed of a flywheel increase from 300 to 600 rev/min in 10 seconds. If the diameter of the wheel is 2 m, determine the angular acceleration and number of revolutions made during this period of 10 seconds. Also workout the normal and tangential acceleration at the rim of the wheel at the end of 10 second. (0.5 rev/s^2, 75 rev, 394 m/s^2, 3.14 m/s^2)

10. In a crank and connecting rod mechanism, the lengths of crank and connecting rod are 15 cm and 65 cm respectively. The crank turns 310 rev/min in the clockwise direction. What will be the velocity of the piston when the crank angle is 60° with respect to the inner dead centre position. (4.7 m/s)

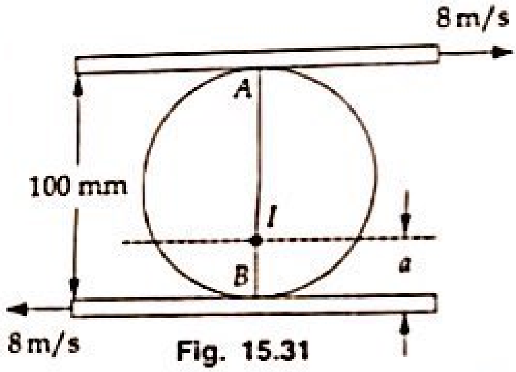
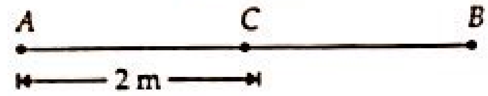


Fig. 15.31

11. A roller of radius 50 mm rides between two horizontal bars moving in opposite directions as shown in Fig. 15.29 Assuming no slip at point of contacts A and B, workout the distance *a* that defines the position of the path of the instantaneous centre of rotation of the roller. (42.86 mm)

12. A horizontal rod *AB* is rotating about the vertical axis through the end *A* at a uniform angular velocity. The linear velocity at the end *B* is 10 m/s and the linear speed of a point *C*, 2 metres from *A*, is 5 m/s. Find the length of the rod and its angular velocity.



Hint: $V_b = \omega \times AB = 10$

$V_c = \omega \times AC = 5$

$\therefore \quad \omega = \frac{5}{AC} = \frac{5}{2} = 2.5 \text{ rad/s}$ and $AB = \frac{10}{\omega} = \frac{10}{2.5} = 4 \text{ m}$

B. Complete the following statements with most appropriate word/words :

1. The shafts, pulleys and flywheels etc., undergo .... motion when they rotate about their .... axis.

2. The linear acceleration *a*, the angular acceleration α and the distance *r* of the body from the centre of rotation are correlated by the identity ....

3. The tangential accelertion is due to change in .... of velocity and the normal acceleration is due to change in .... of the body.

4. A body is said to undergo .... when the particles constituting the rigid body move in parallel planes and travel the same distance.

5. The combination of rotation and translation motion of a body is referred to as .... motion.

6. A rod sliding against a wall at one end and floor at the other end represents a combination of .... motion.

7. .... of the body represents the point about which the plane motion of all the particles constituting the body can be considered as pure rotation.
8. In the absense of external torque, the angular momentum is ....
9. The direction of velocity vector at any point on the circular orbit is .... to the circle at that point.
10. When a bar rotates in horizontal plane, the .... velocity at any point on the bar depends upon its distance from the centre of rotation.

Answers :

1. circular, geometric 2. $a = \alpha r$ 3. magnitude, direction 4. motion of translation 5. general plane motion 6. rotation and translation motion 7. instantaneous centre 8. conserved 9. tangential 10. linear

C. Multiple choice questions :

1. A body moving around a fixed axis constitutes :
   (*a*) curvilinear motion (*b*) circular motion
   (*c*) plane motion (*d*) simultaneous translation and rotation
2. Which of the following units do not conform to the concept of rigid body rotation ?
   (*a*) blades of a fan (*b*) pulley of a belt
   (*c*) phonograph records (*d*) door revolving about a hinge
   (*e*) wheels of an automobile
3. With reference to the adjoining figure, the person A is standing at the centre of a rotating platform facing person B who is riding a bicycle; heading east. The relevant speeds and distances are as given in the figure

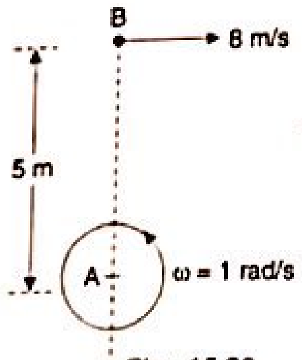


Fig. 15.32.

   At the instant under consideration, the apparent velocity of B as seen by A would be
   (*a*) 3 m/s heading east (*b*) 3 m/s heading west
   (*c*) 8 m/s heading east (*d*) 13 m/s heading east
4. The angular speed of seconds hand of a clock is
   (*a*) π rad/s (*b*) $\pi/6$ rad/s
   (*c*) $\pi/15$ rad/s (*d*) $\pi/30$ rad/s
5. Upon rotation from initially rest position, the radius vector of a body executes angular rotation θ prescribed by the relation $\theta = 4t^3 - 3t^2 + 2t + 6$
   The angular acceleration of the body at time $t = 2$ sec is
   (*a*) 38 rad/s^2 (*b*) 42 rad/s^2
   (*c*) 52 rad/s^2 (*d*) 30 rad/s^2

6. A stone tied to the end of a string 100 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 15 revolutions in 30 seconds, the acceleration of the stone is closest to

(a) 315 cm/s^2
(b) 630 cm/s^2
(c) 985 cm/s^2
(d) 3950 cm/s^2

7. The shaft of a motor starts from rest and attains full speed of 1800 rpm in 10 seconds. The shaft has an angular acceleration of

(a) 3π rad/s^2
(b) 6π rad/s^2
(c) 12π rad/s^2
(d) 24π rad/s^2

8. Two cars are going with constant speeds, round concentric circles of radii r_1 and r_2, and take the same time to complete their circular paths. Their angular speeds will correspond to the ratio

(a) $\sqrt{\frac{r_1}{r_2}}$
(b) $\frac{r_1}{r_2}$
(c) $\left(\frac{r_1}{r_2}\right)^2$
(d) 1

9. The speed of a particle moving in a circle of radius 10 cm changes from 1.5π rad/s to 5π rad/s in 2.2 second time. The corresponding linear acceleration of the particle is

(a) 25 cm/s^2
(b) 32.5 cm/s^2
(c) 50 cm/s^2
(d) 65 cm/s^2

10. Starting from rest, a particle travels on a circular path and the distance covered is prescribed by the relation $s = kt^2$; where k is constant and t is the time. The particle then has a tangential acceleration of

(a) $k/2$
(b) k
(c) $2k$
(d) $4k$

11. An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 km/hr. In terms of acceleration due to gravity ($g = 10$ m/s^2), the centripetal acceleration of aircraft would be

(a) 3.12 g
(b) 6.25 g
(c) 12.5 g
(d) 18.75 g

12. For a particle moving in a circular orbit of radius 0.4 m, the angular velocity and angular acceleration at a particular instant are 2 rad/s and 5 rad/s^2. The particle then has a total linear acceleration of

(a) 1.9 m/s^2
(b) 2.69 m/s^2
(c) 3.8 m/s^2
(d) 7.24 m/s^2

Answers :

1. (b)	2. (d) and (e)	3. (d)	4. (d)	5. (b)	6. (c)
7. (b)	8. (d)	9. (c)	10. (c)	11. (b)	12. (b)